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Abstract

In the category of monoids we characterize monomorphisms that are normal, in an appropriate sense, to internal reflexive relations, preorders or equivalence relations. The zero-classes of such internal relations are first described in terms of convenient syntactic relations associated to them and then through the adjunctions associated with the corresponding normalization functors. The largest categorical equivalences induced by these adjunctions provide equivalences between the categories of relations generated by their zero-classes and the ones of monomorphisms that we suggest to call *normal with respect to* the internal relations considered. This idea, although being transverse to the literature in the field, has not in our opinion been presented and explored in full generality. The existence of adjoints to the normalization functors permits developing a theory of normal monomorphisms, thus extending many results from groups and protomodular categories to monoids and unital categories.

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1. Introduction

The aim of this article is to introduce a relative approach to the concept of a normal subobject in a pointed category. The idea is that a collection of relations determines which subobjects are normal, via a normalization/denormalisation procedure: by definition, *normal* are those subobjects which occur as the zero-class of a relation in the chosen collection. This is a variation of an idea of Bourn in [2] (defining what are normal monomorphisms in protomodular categories) which in the context of monoids, central in this article, leads to some interesting characterizations of some known classes of subobjects which previously were seen as unrelated in a single common framework. In particular, normal subobjects, clots and positive cones are all relative normal subobjects with respect to some class of relations.

In the category of groups every internal reflexive relation is an internal equivalence relation, i.e. a congruence. In addition, the zero-classes of such congruences, which are exactly the normal subgroups, determine all the other classes and so the congruence itself. As it is well known, this is not the case in the category **Mon** of monoids where we have strict inclusions

$$\mathbf{Eq}(\mathbf{Mon}) \subset \mathbf{Ord}(\mathbf{Mon}) \subset \mathbf{RRel}(\mathbf{Mon}) \quad (1)$$

of the full subcategories of the category **Rel(Mon)** of internal relations in monoids consisting of reflexive relations, preorders and equivalence relations. Furthermore, the corresponding zero-classes (that is, the set of elements that are in relation with the zero element) do not determine the relations themselves, in general.

A *clot* [1] is the zero-class of an internal reflexive relation and a *positive cone* is the name of the zero-class of a preorder. A *normal submonoid* is the zero-class of a congruence. In [12], internal transitive and reflexive relations in monoids, i.e. the preordered monoids, denoted there by **OrdMon**, were studied and an equivalence was established between the full subcategory **OrdMon**^{*} of all preordered monoids whose preorder is induced by its positive cone P (where $a \leq_P b \Leftrightarrow b \in P + a$) and the full subcategory of monomorphisms in monoids that consists of what were called there *the right normal monomorphisms* (those inclusions $M \rightarrow A$ for which $a + M \subseteq M + a$ for every $a \in A$). Here we are going to describe the larger equivalence of categories induced by the adjunction $F \dashv N : \mathbf{OrdMon} \rightarrow \mathbf{Mono}(\mathbf{Mon})$, where N is the normalization functor whose left adjoint is described in Proposition 4, which, in particular, provides a categorical definition of positive cone. Indeed, in any pointed category with pullbacks, a positive cone is a normalization of a preorder.

The categorical definition of a clot was given in [9] for semi-abelian categories and then in [10] in pointed regular categories with finite coproducts. In Section 3, we consider clots in what seems to be the most general context where the definition makes sense and their correspondence to internal reflexive relations, called semicongruences there, can be studied.

Inspired by the so-called *syntactic congruence* and *syntactic preorder* for subsets M of a free monoid A (see [4,15]), we define what we call the *M-congruence* and the *M-preorder*, when M is a submonoid of an arbitrary monoid A . The name is justified by the fact that these relations are completely determined by M . Furthermore, they characterize the submonoids of A that are the zero classes of congruences and of preorders, respectively (Proposition 1 and 2).

We also define, for each submonoid M of A , a reflexive relation R_M . Whenever R_M is an internal relation, in which case we call it the *M-reflexive relation* of the submonoid M , we obtain a completely analogous characterization of the submonoids that are clots (Proposition 3). There is a large class of monoids (which includes the finite and the Dedekind finite monoids)

for which these reflexive relations are indeed internal. However, it is an open problem to decide whether this holds or not for all monoids as stated in Example J (see Section 4).

So natural (and useful) these relations may be, they are not functorial: there is no functor with these object functions as we show in Example H. In contrast, in Section 3, we describe such functorial relations as left adjoints to the three normalization functors

$$N : \mathcal{R} \rightarrow \mathbf{Mono}(\mathbf{Mon}),$$

where $\mathcal{R}$ is one of the categories of internal relations referred to above as instances of a more general situation (Theorem 1).

The largest equivalence of categories induced by each of these adjunctions $F \dashv N(\eta, \varepsilon)$, i.e. $Fix(\varepsilon) \sim Fix(\eta)$, between the full subcategories with all the objects for which the counit or the unit are isomorphisms, respectively, have a nice interpretation. In each case, the full subcategory $Fix(\varepsilon)$ consists of all the corresponding relations that are generated by the zero-class (in the sense that such a relation is the smallest one with the same zero-class) and $Fix(\eta)$ is the full subcategory $\mathbf{NMono}(\mathbf{Mon})$ of $\mathbf{Mono}(\mathbf{Mon})$ consisting of what we call *the normal monomorphisms* with respect to objects of $\mathcal{R}$ that are exactly the clots, the positive cones and the normal monomorphisms, respectively.

Organization of this article. In Section 2, specific congruences, internal preorders and reflexive relations on a monoid A , associated with each submonoid M of A , are introduced in order to characterize their zero-classes. In Section 3, we define left adjoints to the corresponding normalization functors as a specialization of a more general situation in the context of a pointed, wellpowered and finitely complete category $\mathbf{C}$ with arbitrary intersections of subobjects and subcategories $\mathcal{R}$ of binary relations of a fixed type on objects of $\mathbf{C}$, satisfying some conditions. In a pointed variety $\mathcal{V}$, the smallest internal relation on an object A of $\mathcal{V}$ whose zero class contains a subobject M of A can be described as a special subobject of $A \times A$, as we do here when $\mathcal{V} = \mathbf{Mon}$, providing an alternative way to describe these left adjoints, a straightforward procedure that contributes to better exhibiting the differences between the corresponding zero-classes. We gather our observations in three theorems that, in particular, give a systematic way to construct for each monomorphism that is normal to one of the internal relations considered above the smallest internal relation to which it is normal to. In the last section, we present several examples illustrating our previous claims.

2. Syntactic relations of monoids

In this section we define binary relations from the syntactical point of view that will play a role in the characterizations of clots, positive cones and normal submonoids presented in Theorems 2–4.

The syntactic monoid is widely used in computation theory due to its applications in automata theory and regular languages (see [4,8,13,15]). According to Eilenberg ([4], p.75), the first clear-cut and systematic exposition of the syntactic monoid can be found in [13]. The syntactic congruence and syntactic preorder are usually defined for a subset M of a free monoid Σ^* . For our purposes we will consider an arbitrary monoid A instead of a free monoid and a submonoid M of A rather than a subset. However, both the syntactic congruence and the syntactic preorder are internal relations regardless of whether M is a submonoid or just a subset of A . The relevance of this aspect in our study contrasts with the one usually needed in computer science. Instead of asking whether or not a subset M of a monoid A is recognizable (in the sense of [4], p. 68), we will be interested in knowing whether or not M is the zero-class of the syntactic congruence or of the syntactic preorder, in which cases M is necessarily a submonoid of A . For some examples, see H, I, J and K in Section 4.

2.1. The syntactic congruence

Given a subset M of a monoid A , the syntactic M -congruence is defined by $a \sim_M b$ if, for every $x, y \in A$,

$$xay \in M \Leftrightarrow xby \in M.$$

This binary relation is an equivalence relation. Furthermore, it is compatible with the monoid operation: if $a \sim_M b$ and $a' \sim_M b'$ then, for every $x, y \in A$,

$$xa(a'y) \in M \Leftrightarrow xb(a'y) = (xb)a'y \in M \Leftrightarrow (xb)b'y \in M.$$

In other words, the set $E = \{(a, b) \in A \times A \mid a \sim_M b\}$ is a submonoid of $A \times A$, and being an equivalence relation, it is a congruence on A . The syntactic monoid is the quotient A/E and the zero-class of E ,

$$[1]_E = \{u \in A \mid (1, u) \in E\},$$

is a submonoid of A .

Proposition 1. *Let M be a submonoid of the monoid A . Then M is the zero-class of some congruence in A if and only if the following condition holds*

(C) *For every $x, y \in A$ and every $u \in M$,*

$$xy \in M \Leftrightarrow xuy \in M.$$

In this case, M is exactly the zero class of $\sim_M$.

Proof. The zero class of $\sim_M$ is contained in M . Indeed, if $1 \sim_M u$ then, for every $x, y \in A$,

$$xy \in M \Leftrightarrow xuy \in M.$$

So, in particular, for $x = y = 1$, $1 \in M \Leftrightarrow u \in M$.

Furthermore, $M \subseteq [1]_{\sim_M}$ whenever (C) holds: if $u \in M$ then it belongs to the zero class of the M -congruence when

$$xy \in M \Leftrightarrow xuy \in M,$$

that is exactly condition (C).

If $M = [1]_{\sim}$ for some congruence $\sim$ on A then if $1 \sim u$, by reflexivity and compatibility, we conclude that $xy \sim xuy$. Since $xy \in M$ means that $1 \sim xy$ then also $1 \sim xuy$, by transitivity of the relation, and so $xuy \in M$. Consequently, condition (C) is fulfilled.

So, we conclude that M is the zero class for some congruence on A if and only if it is the zero class of its M -congruence. $\square$

Remark 1. In [5] Facchini and Rodaro present (Thm 9), the following necessary and sufficient condition for a submonoid M of a monoid A to be a kernel:

(F) For every $x, y \in A$,

$$xMy \cap M \neq \emptyset \Rightarrow xMy \subseteq M,$$

that is, obviously, equivalent to condition

(C) For every $x, y \in A$ and every $u \in M$,

$$xy \in M \Leftrightarrow xuy \in M,$$

as it is easy to prove:

(F) $\Rightarrow$ (C)

If $x, y \in A$ and $u \in M$ then

$$xuy \in M \Rightarrow xMy \cap M \neq \emptyset \Rightarrow xMy \subseteq M \Rightarrow xuy \in M.$$

And

$$xuy \in xMy \cap M \Rightarrow xMy \subseteq M \Rightarrow x1y = xy \in M.$$

(C) $\Rightarrow$ (F)

For $x, y \in A$, suppose that there exists an element $u \in M$ such that $xuy \in M$. Since we know that for every $v \in M$,

$$xuy \in M \Leftrightarrow xy \in M \Leftrightarrow xvy \in M,$$

we deduce $xMy \subseteq M$.

Note that kernels need not coincide with normal subobjects in non-exact categories like for example the category of torsion free abelian groups (see e.g. [7]).

2.2. The syntactic preorder

Every subset M of a monoid A induces an internal reflexive and transitive relation $\leq_M$ on A called the syntactic preorder (see [15]). It is defined as $a \leq_M b$ if for every $x, y \in A$

$$xay \in M \Rightarrow xby \in M.$$

This binary relation is reflexive, transitive and compatible with the monoid operation. Its positive cone consists of all $a \in A$ such that $1 \leq_M a$, which means that every $x, y \in A$,

$$xy \in M \Rightarrow xay \in M,$$

implying that it is a submonoid of A .

Proposition 2. *Let M be a submonoid of a monoid A . Then M is the positive cone of some preorder on A if and only if the following condition holds*

(P) *For every $x, y \in A$ and every $u \in M$,*

$$xy \in M \Rightarrow xuy \in M.$$

In this case, M is the positive cone of $\leq_M$.

Proof. The proof is analogous to the one of the previous proposition and, in the same way, we conclude that M is the positive cone of some preorder on A if and only if it is the positive cone of its associated M -preorder. $\square$

Example H in Section 4 shows that neither of the above syntactic constructions is functorial. Let us now investigate the case of clots, that is the zero-classes of internal reflexive relations.

2.3. The syntactic reflexive relation

Every submonoid M of a monoid A induces a reflexive relation R_M in A defined by aR_Mb if for every $x, y \in A$,

$$xay = 1 \Rightarrow xby \in M.$$

As we show next there is a large class of monoids where R_M is an internal reflexive relation, in which case we call it the syntactic M -reflexive relation of the submonoid M . We point out that we do not know if this is the case for all monoids.

Proposition 3. *Let M be a submonoid of a monoid A . Then M is the zero-class of the reflexive relation R_M if and only if the following condition holds*

(R) *For every $x, y \in A$ and every $u \in M$*

$$xy = 1 \Rightarrow xuy \in M.$$

If M is a clot then $M = [1]_{R_M}$ and the converse holds when the relation R_M is internal.

Proof. Like in the previous cases, we have that $[1]_{R_M}$ is a subset of M and that $M \subseteq [1]_{R_M}$ if and only if condition (R) is satisfied. Consequently, in presence of condition (R), we have that $M = [1]_{R_M}$.

Let M be the zero class of some internal reflexive relation S on A . If $1Su$, by the reflexivity and compatibility of the relation, $xySxuy$. Then, if $xy = 1$ we have that $1Sxuy$, that is $xuy \in M$, (R) holds and so $M \subseteq [1]_{R_M}$.

The converse is true if R_M is compatible. $\square$

The relation R_M is compatible with the monoid operation as soon as the following extra condition is satisfied

$$\forall x, y, s, t \in A, (xy = 1, xs \in M, ty \in M) \Rightarrow ts \in M \quad (2)$$

It is for instance implied by the condition saying that every left (or right) invertible element in A is invertible:

$$\forall x, y \in A, (xy = 1 \Rightarrow yx = 1) \quad (3)$$

that characterizes the Dedekind finite monoids, also called Von Neumann finite or directly finite monoids (see [6]). Dedekind finite monoids include groups, commutative monoids, finite monoids, cancellative monoids and others, but not all monoids. For example, for any infinite set X , the monoid $\mathbf{Set}(X, X)$ with respect to composition of functions as well as the bicyclic monoid are not Dedekind finite.

In Dedekind finite monoids from $xs, ty \in M$ we get $tyxs \in M$ and, if $yx = 1$ as soon as $xy = 1$, then we conclude $ts \in M$.

The condition of being Dedekind finite is stronger than Condition (2) which is in turn strictly stronger than compatibility of R_M as shown in Example I (see Section 4).

Remark 2. In the category of groups, condition (R) means that M is a normal subgroup of the group A and the congruence R_M coincides with the classical one induced by the normal subgroup M , that is

$$a \sim b \Leftrightarrow ab^{-1} \in M.$$

We recall that $\sim$ is an equivalence relation on the underlying set of A if and only if M is a subgroup and that it is a congruence if and only if M is a normal subgroup of A .

3. Left adjoints to the normalization functors

We are going to describe the left adjoints $F: \mathbf{Mono}(\mathbf{Mon}) \rightarrow \mathcal{R}$ to the corresponding normalization functors, as examples of a construction in a more general context, which we make first.

Let $\mathbf{C}$ be a pointed, well-powered and finitely complete category with arbitrary intersections, that is with generalized pullbacks of monomorphisms.

By $\mathcal{R} \subseteq \mathbf{Rel}(\mathbf{C})$ we denote the category of internal binary relations of a fixed type on $\mathbf{C}$ -objects with morphisms the morphisms in $\mathbf{C}$ compatible with the relations. In addition, we assume that such relations are stable under pullback and closed under arbitrary intersections for every object of $\mathbf{C}$, that is, we assume that the functor $(\)_0: \mathcal{R} \rightarrow \mathbf{C}$ that assigns to each relation its base object is a fibration and that each fiber $\mathcal{R}_A$ is a complete lattice. So, in particular, we assume that, for each object A in $\mathbf{C}$, the indiscrete relation ∇_A belongs to $\mathcal{R}$.

Definition 1. A monomorphism $m: M \rightarrow A$ is said to be normal w.r.t. $\mathcal{R}$ if it is part of a pullback in $\mathbf{C}$

$$\begin{array}{ccc} M & \xrightarrow{u} & R \\ m \downarrow & & \downarrow r \\ A & \xrightarrow{(0,1)} & A \times A \end{array}$$

for a relation (R, r) in $\mathcal{R}$ on A .

It is easy to prove that such morphisms are stable under pullback when the same holds for the relations in $\mathcal{R}$. The above definition includes clots, positive cones and normal subobjects in the classical sense in any pointed variety but makes sense in every non-varietal category satisfying the prescribed conditions, defining there the full subcategory $\mathbf{NMono}(\mathbf{C}, \mathcal{R})$ of the category $\mathbf{Mono}(\mathbf{C})$ consisting of what we call the *normal monomorphisms with respect to $\mathcal{R}$* .

We point out that the categorical notion of normal monomorphism with respect to a congruence was introduced in [2].

The normalization functor

$$N: \mathcal{R} \rightarrow \mathbf{Mono}(\mathbf{C})$$

is defined on objects by $N(A, R) = m$ as in the diagram of Definition 1. Indeed, if $r_i = p_i \cdot r$ with p_i the direct product projections for $i = 1, 2$, then u is the kernel of r_1 and $m = r_2 \cdot u$, the classical definition of the normalization functor [2]. The universal property of pullbacks provides the definition of N on morphisms.

Theorem 1. Let $\mathbf{C}$ be a pointed, well-powered and finitely complete category with arbitrary intersections and $\mathcal{R}$ be a pullback-stable full subcategory of the one of internal relations on $\mathbf{C}$ closed under arbitrary intersections. Then the normalization functor $N: \mathcal{R} \rightarrow \mathbf{Mono}(\mathbf{C})$ has a left adjoint, called the denormalization functor,

$$F: \mathbf{Mono}(\mathbf{C}) \rightarrow \mathcal{R}$$

defined on objects by $F(m: M \rightarrow A) = (R, r)$ with $(R, r) = \bigcap \mathbf{S}_m$, where $\mathbf{S}_m$ is the set of all relations in $\mathcal{R}$ on A through which $\langle 0, 1 \rangle m$ factors.

Proof. We define $F(m) = (R, r) = \bigcap S_m$, where S_m is not empty because it contains ∇_A since $\mathcal{R}$ is closed under arbitrary intersections.

The fact that the relations in $\mathcal{R}$ on $\mathbf{C}$ are stable under pullback, enables us to assign to each $(f, \bar{f}): m \rightarrow m'$ a morphism

$$(f \times f, g): F(m) = (R, r) \rightarrow F(m') = (R', r')$$

where $g: R \rightarrow R'$ is the morphism $t \cdot l$, $(t, \bar{r}')$ being the pullback of r' along $f \times f$, which gives the relation $(\bar{R}', \bar{r}')$ that belongs to S_m and so there exists a unique morphism l such that $\bar{r}' \cdot l = r$:

$$\begin{array}{ccccc} R & \xrightarrow{l} & \bar{R}' & \xrightarrow{t} & R' \\ r \downarrow & & \downarrow \bar{r}' & & \downarrow r' \\ A \times A & \xlongequal{\quad} & A \times A & \xrightarrow{f \times f} & A' \times A' \end{array} \quad (4)$$

This defines a functor F which, in addition, is left adjoint of N with unit $\eta_m = (1_A, s): m \rightarrow NF(m)$, the normal closure of m ,

$$\begin{array}{ccc} M & \xrightarrow{u} & R \\ \downarrow m & \searrow s & \uparrow \\ & P & \\ \downarrow NF(m) & \nearrow & \downarrow F(m) \\ A & \xrightarrow{\langle 0, 1 \rangle} & A \times A \end{array} \quad (5)$$

where P is the pullback of $F(m)$ along $\langle 0, 1 \rangle$. $\square$

In the case of congruences in Barr-exact categories there is another way to describe the left adjoint to the normalization functor.

Corollary 1. *Let $\mathbf{C}$ be a Barr-exact category and $\mathcal{R}$ denote the category $\mathbf{Eq}(\mathbf{C})$ of internal equivalence relations on $\mathbf{C}$. Then $F(m)$ is the kernel pair of the cokernel of m .*

Proof. If $F(m) = (R, r)$ and (R, r_1, r_2) is the kernel pair of some morphism in $\mathbf{C}$ then it is the kernel pair of the coequalizer $q: A \rightarrow B$ of $r_1, r_2: R \rightarrow A$ and so, up to isomorphism, (R, r_1, r_2) coincides with $(A \times_B A, \pi_1, \pi_2)$.

It remains to prove that q is the cokernel of m , that is, it is the coequalizer of $(m, 0)$. If $fm = f0$ then we have that the kernel pair of f belongs to S_m and so we conclude that $fr_1 = fr_2$. Consequently, there is a unique morphism f' such that $f'q = f$. $\square$

Recall that an internal binary relation (R, r_1, r_2) on an object A of $\mathbf{C}$ is

- *reflexive* if there exists a morphism $e: A \rightarrow R$ such that $r_1 e = r_2 e = 1_A$, that is, if $1_A \leq R$.
- *symmetric* if there exists a morphism $s: R \rightarrow R$ such that $r_1 s = r_2$ and $r_2 s = r_1$, that is, if $R^o \leq R$.
- *transitive* if there exists a morphism $t: R \times_A R \rightarrow R$ such that $r_1 t = r_1 \pi_1$ and $r_2 t = r_2 \pi_2$, that is, if $RR \leq R$ whenever $\mathbf{C}$ is a regular category.

If, in [Theorem 1](#), $\mathcal{R}$ denotes the category of reflexive relations, preorders or equivalence relations in $\mathbf{C}$ (but not the category of all orders, because the only set A for which ∇_A is an order is a singleton or the empty set) then $\mathcal{R}$ satisfy the prescribed conditions and so we obtain adjunctions

$$\mathbf{RRel}(\mathbf{C}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{R} \end{matrix} \mathbf{Mono}(\mathbf{C}) \quad (6)$$

$$\mathbf{Ord}(\mathbf{C}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{P} \end{matrix} \mathbf{Mono}(\mathbf{C}) \quad (7)$$

$$\mathbf{Eq}(\mathbf{C}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{E} \end{matrix} \mathbf{Mono}(\mathbf{C}) \quad (8)$$

denoting the functor F by R , P and E , respectively. This is the case when $\mathbf{C} = \mathbf{Mon}$.

The above adjunctions are restrictions of the general adjunction

$$\mathbf{Rel}(\mathbf{C}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{E} \end{matrix} \mathbf{Mono}(\mathbf{C}) ,$$

since $\mathcal{R} = \mathbf{Rel}(\mathbf{C})$ satisfies the conditions of [Theorem 1](#).

Whenever $\mathbf{C}$ is a pointed variety we have another classical procedure to define these left adjoints that consists of assigning to each monomorphism $m: M \rightarrow A$, that, without lost of generality, we assume to be an inclusion, the smallest zero-class in A that contains M , by describing appropriate submonoids of $A \times A$. The explicit descriptions of such submonoids, that we present next, enable us to better exhibit different aspects of the zero classes of each of the internal relations as shown in [Example B](#) of [Section 4](#).

In the rest of this section, for simplicity of exposition, we use additive notation but we do not assume that the monoids are commutative.

By $R = R(m)$ we denote the smallest internal reflexive relation on a monoid A whose zero class contains M , that is, the submonoid of $A \times A$ generated by the set $(\{0\} \times M) \cup \Delta_A$. This means that xRy if and only if there exists a natural number $n \in \mathbb{N}$ together with sequences $a_i \in A$ and $u_i \in M$, with $i = 1, \dots, n$, such that

$$x = a_1 + \dots + a_n \text{ and } y = a_1 + u_1 + \dots + a_n + u_n. \quad (9)$$

The relation $P = P(m)$ is the smallest reflexive and transitive relation (a preorder) on A whose zero class contains all the elements in M . In this case it is the transitive closure of the above internal reflexive relation which is again internal, as it is easy to check. We conclude that xPy if and only if there exist natural numbers $m, n \in \mathbb{N}$ together with double indexed sequences $a_{ij} \in A$ and $u_{ij} \in M$, with $i = 1, \dots, m$ and $j = 1, \dots, n$, such that

$$\begin{aligned} x &= \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n (a_{1j} + u_{1j}) &= \sum_{j=1}^n a_{2j} \\ &\dots \\ \sum_{j=1}^n (a_{(i-1)j} + u_{(i-1)j}) &= \sum_{j=1}^n a_{ij} \end{aligned}$$

...

$$\sum_{j=1}^n (a_{(m-1)j} + u_{(m-1)j}) = \sum_{j=1}^n a_{mj}$$

$$\sum_{j=1}^n (a_{mj} + u_{mj}) = y.$$

The relation $E = E(m)$ is the smallest internal equivalence relation on A whose zero class contains M . We start by considering the submonoid of $A \times A$ generated by $(M \times M) \cup \Delta_A$ that is the smallest internal reflexive and symmetric relation whose zero-class contains M . Then the transitive closure of this relation gives E . Thus we have that xEy if and only if there exist natural numbers $m, n \in \mathbb{N}$ together with double indexed sequences $a_{ij} \in A$ and $u_{ij}, v_{ij} \in M$, with $i = 1, \dots, m$ and $j = 1, \dots, n$, such that

$$x = \sum_{j=1}^n (a_{1j} + v_{1j})$$

$$\sum_{j=1}^n (a_{1j} + u_{1j}) = \sum_{j=1}^n (a_{2j} + v_{2j})$$

...

$$\sum_{j=1}^n (a_{(i-1)j} + u_{(i-1)j}) = \sum_{j=1}^n (a_{ij} + v_{ij})$$

...

$$\sum_{j=1}^n (a_{(m-1)j} + u_{(m-1)j}) = \sum_{j=1}^n (a_{mj} + v_{mj})$$

$$\sum_{j=1}^n (a_{mj} + u_{mj}) = y.$$

Proposition 4. *If we denote by N the normalization functor from the categories of internal reflexive relations, preorders and equivalence relations in the category **Mon** of monoids, we have the following adjunctions*

$$\mathbf{RRel}(\mathbf{Mon}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{R} \end{matrix} \mathbf{Mono}(\mathbf{Mon}) \quad (10)$$

$$\mathbf{Ord}(\mathbf{Mon}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{P} \end{matrix} \mathbf{Mono}(\mathbf{Mon}) \quad (11)$$

$$\mathbf{Eq}(\mathbf{Mon}) \begin{matrix} \xrightarrow{N} \\ \xleftarrow{E} \end{matrix} \mathbf{Mono}(\mathbf{Mon}) \quad (12)$$

where the functors R , P and E are defined on objects as described above.

The largest equivalence $\mathbf{Fix}(\varepsilon) \sim \mathbf{Fix}(\eta)$ induced by each of these adjunctions, consists of $\mathbf{Fix}(\varepsilon)$, the category of all relations that are generated by their zero-classes in the sense

that they are the smallest relations with these zero-classes, and $\mathbf{Fix}(\eta)$, the category of all the corresponding zero-classes. In the three particular cases of our study, $\mathbf{Fix}(\eta)$ consist of all clots, all positive cones and all normal submonoids.

Let us denote by $\mathbf{Fix}(\eta_R)$, $\mathbf{Fix}(\eta_P)$ and $\mathbf{Fix}(\eta_E)$ the full subcategories of $\mathbf{Mono}(\mathbf{Mon})$ consisting of all monomorphisms $m: M \rightarrow A$ such that the corresponding adjunction unit η_m is an isomorphism. Similarly, let us denote by $\mathbf{Fix}(\varepsilon_E)$, $\mathbf{Fix}(\varepsilon_P)$ and $\mathbf{Fix}(\varepsilon_R)$ the respective subcategories of relations that are determined by their respective zero-classes.

The following theorems sum up our observations with respect to normal monomorphism relative to the internal binary relations we are dealing with in $\mathbf{Mon}$. There, for every monomorphism $m: M \rightarrow A$ we always identify $m(M)$ with M .

Theorem 2 (Normal Subobjects). *A monoid monomorphism $m: M \rightarrow A$ is a normal subobject if one, and so all, of the following equivalent conditions is satisfied:*

- (a) M is the zero class of some congruence;
- (b) M is the zero-class of its syntactic congruence;
- (c) M is the zero-class of the congruence $E(m)$;
- (d) M is the zero-class of the smallest congruence whose zero-class contains M ;
- (e) m belongs to $\mathbf{Fix}(\eta_E)$, that is, m is normal w.r.t. $\mathbf{Eq}(\mathbf{Mon})$;
- (f) $E(m)$ belongs to $\mathbf{Fix}(\varepsilon_E)$;
- (g) for every $x, y \in A$ and every $u \in M$,

$$xy \in M \Leftrightarrow xuy \in M;$$

- (h) for every natural numbers $m, n \in \mathbb{N}$ and all double indexed sequences $a_{ij} \in A$, $u_{ij}, v_{ij} \in M$, with $i = 1, \dots, m$ and $j = 1, \dots, n$, if

$$0 = a_{11} + v_{11} + \dots + a_{1n} + v_{1n}$$

and

$$a_{11} + u_{11} + \dots + a_{1n} + u_{1n} = a_{21} + v_{21} + \dots + a_{2n} + v_{21}$$

...

$$a_{(i-1)1} + u_{(i-1)1} + \dots + a_{(i-1)n} + u_{(i-1)n} = a_{i1} + v_{i1} + \dots + a_{in} + v_{in}$$

...

$$a_{(m-1)1} + u_{(m-1)1} + \dots + a_{(m-1)n} + u_{(m-1)n} = a_{m1} + v_{m1} + \dots + a_{mn} + v_{mn}$$

then

$$a_{m1} + u_{m1} + \dots + a_{mn} + u_{mn} \in M.$$

Theorem 3 (Positive Cones). *A monoid monomorphism $m: M \rightarrow A$ is a positive cone if one, and so all, of the following equivalent conditions is satisfied:*

- (a) M is the zero-class of some preorder;
- (b) M is the zero-class of its syntactic M -preorder;
- (c) M is the zero-class of the preorder $P(m)$;
- (d) M is the zero-class of the smallest preorder whose zero-class contains M ;
- (e) m belongs to $\mathbf{Fix}(\eta_P)$, that is, m is normal w.r.t. $\mathbf{Ord}(\mathbf{Mon})$;
- (f) $P(m)$ belongs to $\mathbf{Fix}(\varepsilon_P)$;

(g) for every $x, y \in A$ and every $u \in M$,

$$xy \in M \Rightarrow xuy \in M;$$

(h) for every natural numbers $m, n \in \mathbb{N}$ and all double indexed sequences $a_{ij} \in A$, $u_{ij} \in M$, with $i = 1, \dots, m$ and $j = 1, \dots, n$, if

$$0 = a_{11} + \dots + a_{1n}$$

and

$$a_{11} + u_{11} + \dots + a_{1n} + u_{1n} = a_{21} + \dots + a_{2n}$$

...

$$a_{(i-1)1} + u_{(i-1)1} + \dots + a_{(i-1)n} + u_{(i-1)n} = a_{i1} + \dots + a_{in}$$

...

$$a_{(m-1)1} + u_{(m-1)1} + \dots + a_{(m-1)n} + u_{(m-1)n} = a_{m1} + \dots + a_{mn}$$

then

$$a_{m1} + u_{m1} + \dots + a_{mn} + u_{mn} \in M.$$

Theorem 4 (Clots). A monoid monomorphism $m: M \rightarrow A$ is a clot if one, and so all, of the following equivalent conditions is satisfied:

- (a) M is the zero-class of some internal reflexive relation;
- (b) M is the zero-class of the reflexive relation $R(m)$;
- (c) M is the zero-class of the smallest internal reflexive relation whose zero-class contains M ;
- (d) m belongs to $\mathbf{Fix}(\eta_R)$, that is, m is normal w.r.t. $\mathbf{RRel}(\mathbf{Mon})$;
- (e) $R(m)$ belongs to $\mathbf{Fix}(\varepsilon_R)$;
- (f) for every natural number $n \in \mathbb{N}$ and for every two sequences $a_i \in A$, $u_i \in M$, with $i = 1, \dots, n$, if

$$a_1 + \dots + a_n = 0$$

then

$$a_1 + u_1 + \dots + a_n + u_n \in M.$$

Furthermore, when R_M is an internal relation (i.e., it is the syntactic M -reflexive relation) the above conditions are also equivalent to the following:

- (g) M is the zero-class of its syntactic reflexive relation;
- (h) for every $x, y \in A$ and every $u \in M$,

$$xy = 1 \Rightarrow xuy \in M.$$

4. Examples

Examples A–C illustrate the strict inclusions displayed in (1). Example D describes a normal submonoid which is not a subgroup. Examples E–K are related with observations made in Section 2.

- (A) A simple example of a submonoid that is not a clot is obtained as follows. Take the monoid with four elements $\{1, 2, 3, 4\}$ and multiplication defined by

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	2	3	3
3	3	2	3	2
4	4	2	3	1

(13)

and observe that $M = \{1, 2\}$ is a submonoid. However, the smallest compatible reflexive relation R containing the pair $(1, 2)$ is

$$\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$$

whose zero-class is the submonoid $N(R) = \{1, 2, 3\}$.

- (B) An example of a clot which is not a positive cone is provided by the monomorphism $m: \mathbf{N} \rightarrow F(a, b)$, from the natural numbers to the free monoid on two generators a and b , which associates $1 \in \mathbf{N}$ to the word ab . Then $m(n) = (ab)^n$ and $\mathbf{N}$ is the clot of a reflexive relation but it is not the positive cone of any reflexive and transitive relation on $A = F(a, b)$.

We can prove that in two different ways:

- (i) First we observe that in $R(m)$ we have that $1Rx$ if and only if $x = (ab)^n$ for some natural number n . This already shows that $\mathbf{N}$ is a clot. But it is not a positive cone because the relation is not transitive. Indeed, otherwise, $1Raabb$ should hold since $1Rab$ and $abRaabb$, but $aabb$ is not of the form $(ab)^n$ for any natural n . The fact that $abRaabb$ follows from reflexivity and compatibility of the relation: since aRa , $1Rab$ and bRb we obtain $abRaabb$.
 - (ii) Another way to see that is to use the characterizations given in [Theorems 4](#) and [3](#). First we observe that condition (f) of [Theorem 4](#) is satisfied because $a_1 + \dots + a_n = 0$ implies that all a_i have to be zero. Secondly, we observe that condition (h) of [Theorem 3](#) is not satisfied: taking $m = 2$, $n = 2$, $a_{1j} = 0$, $u_{11} = (ab)$, $u_{12} = 0$, $a_{21} = (a)$, $a_{22} = (b)$, $u_{21} = (ab)$, $u_{22} = 0$ we observe that $(ab) = (a) + (b)$ but $(a) + (ab) + (b) = aabb$ is not of the form $(ab)^n$ and hence it is not in the image of the monomorphism $m: \mathbf{N} \rightarrow F(a, b)$.
- (C) An example of a positive cone which is not a normal submonoid (with respect to **Eq(Mon)**) is provided by the inclusion of the natural numbers in the group of the integers.
- (D) The inclusion of the even numbers in the monoid of the natural numbers is an example of a normal submonoid. Indeed it satisfies the condition (g) of [Theorem 2](#).
- (E) For every finite monoid, the relation R_M is an internal reflexive relation ([Section 2](#)) because these monoids are Dedekind finite.
- (F) If X is an infinite set, the monoid **Set**(X, X) of endofunctions of X for composition is not Dedekind finite. Indeed any surjective function that is not injective has a right inverse but not a left inverse.
- (G) The bicyclic monoid is the monoid freely generated by two elements b and c , satisfying the equation $bc = 1$. It is not Dedekind finite since $bc = 1$ but $cb \neq 1$.

- (H) There is no functor $F: \mathbf{Mono}(\mathbf{Mon}) \rightarrow \mathbf{Eq}(\mathbf{Mon})$ assigning to each submonoid M of A its syntactic M -congruence (Section 2). To show that let us consider the composition of morphisms

$$S_2 \xrightarrow{f} S_3 \xrightarrow{g} S_2$$

of symmetric groups such that $gf = 1_{S_2}$. It induces in $\mathbf{Mono}(\mathbf{Mon})$ the identity of the monomorphism $m = 1_{S_2}$, that is, $(f, 1_{S_2}) \circ (g, 1_{S_2}) = (1_{S_2}, 1_{S_2})$ as in the following diagram

$$\begin{array}{ccc} S_2 & \xrightarrow{m} & S_2 \\ \parallel & & \downarrow f \\ S_2 & \xrightarrow{m'=f} & S_3 \\ \parallel & & \downarrow g \\ S_2 & \xrightarrow{m} & S_2. \end{array}$$

There is no way to define an arrow function for F sending this composition to the identity of $F(m) = (S_2, \nabla_{S_2})$. Indeed, it is enough to notice that the zero-class of $\sim_{S_2}$ in S_3 is the trivial group $\{\epsilon\}$ because it is contained in S_2 and it does not contain (12):

$$(13)\epsilon(13) = \epsilon \in S_2 \text{ but } (13)(12)(13) = (23) \notin S_2.$$

So, $\sim_{S_2}$ is equal to Δ_{S_3} . Then no composition of morphisms

$$(S_2, \nabla_{S_2}) \xrightarrow{(f', f' \times f')} (S_3, \Delta_{S_3}) \xrightarrow{(g', g' \times g')} (S_2, \nabla_{S_2})$$

could be the identity of (S_2, ∇_2) , since f' has to send (12) to ϵ and so it is the zero morphism.

Consequently, there is no functor sending a submonoid M of A to its M -preorder or to its M -reflexive relation.

- (I) For every submonoid M of A we have that if A is Dedekind finite then condition (2) holds and that if A satisfies (2) then R_M is compatible but these two implications are strict. Indeed,

- (i) If $M = A$ then condition (2) holds and A may be one of the examples of a non-Dedekind finite monoid.
- (ii) There are internal reflexive relations R_M that do not satisfy condition (2). Let $A = \mathbf{Set}(\mathbb{N}, \mathbb{N})$ for composition of functions and $M = \{1_{\mathbb{N}}\}$. Then R_M is compatible since now $aR_M b$ if, for all functions $x, y: \mathbb{N} \rightarrow \mathbb{N}$,

$$xay = 1 \Rightarrow xby = 1,$$

and we can find examples of functions x, y, s, t such that $xy = 1, xs = 1$ and $ty = 1$ but ts is not equal to 1. For example,

$x: \mathbb{N} \rightarrow \mathbb{N}$ taking zero to zero and $n > 0$ to $n - 1$;

$y: \mathbb{N} \rightarrow \mathbb{N}$ sending each $n > 0$ to $n + 1$;

$s: \mathbb{N} \rightarrow \mathbb{N}$ taking zero to zero and $n > 0$ to $n + 1$;

$t: \mathbb{N} \rightarrow \mathbb{N}$ taking 0 to 5 and $n > 0$ to $n - 1$.

Then $xy = 1, xs = 1$ and $ty = 1$ but $ts(0) = 5$.

- (J) We do not know whether or not R_M is always an internal relation. The following example brings some light on the problem but does not solve it.

Let A be the monoid $\mathbf{Set}(\mathbb{N}, \mathbb{N})$, of all endofunctions of the natural numbers for composition, and M be the submonoid generated by the function $u = 2 \times -$. Then M consists of all functions $u^n = 2^n \times -$ for some $n \in \mathbb{N}_0$.

Then u^n for $n > 0$ does not belong to the zero set of the relation R_M . Indeed, for $f, g \in A$ defined by $g(x) = x + 1$ and $f(x) = x - 1$ if $x > 1$ and $f(1) = 1$ we have that $f \cdot g = 1_{\mathbb{N}}$ but $f \cdot u^n \cdot g \notin M$ because $f \cdot u^n \cdot g(x) = 2^n(x + 1) - 1$, for $x \in \mathbb{N}$.

Thus, $[1]_{R_M} = \{1_{\mathbb{N}}\}$ and so M is not the zero class of any internal reflexive relation on A .

- (K) For any submonoid M of a monoid A we have the following inclusions

$$[1]_{R_M} \subseteq M \subseteq [1]_{R(m)},$$

that, in the previous example, are both strict. Indeed, not only $[1]_{R_M}$ is a proper submonoid of M but also

$$f \cdot u^n \cdot g \in [1]_{R(m)}$$

because $1 = f \cdot g$ and $h = f \cdot u^n \cdot g$ and so $1R(m)h$ but h does not belong to M .

5. Final remarks

In [9], the authors give a categorical definition of clot in semi-abelian categories that they extend in [10] to the context of pointed regular categories with finite coproducts. They prove that clots are the same as zero classes of internal reflexive relations, called there semicongruences. In this work we consider clots in the context of pointed, well-powered and finitely complete categories with arbitrary intersections of subobjects that seems to be the most general context where studying clots versus internal reflexive relations makes sense. In particular, if $\mathbf{C} = \mathbf{Mon}$, the semicongruence associated to each clot $m: M \rightarrow A$ in Theorem 2.2 of [10] is the internal relation $R(m)$ defined above.

In [14], for regular categories $\mathbf{C}$ with pushouts of split monomorphisms along arbitrary maps, G. Metere proves that the restriction of the bifibration $(\)_0: \mathbf{RGraph}(\mathbf{C}) \rightarrow \mathbf{C}$ to $\mathbf{RRel}(\mathbf{C})$ is a bifibration and this, in particular, enables him to construct a left adjoint to the normalization functor. When $\mathbf{C}$ is the category of monoids, this gives another way to construct the left adjoint (10) and, in [3], for Mal'tsev categories, this provides the left adjoint to $N: \mathbf{Eq}(\mathbf{C}) \rightarrow \mathbf{Mono}(\mathbf{C})$.

In order to cover ideals, as a generalization of clots [1,11], we would need to replace the study of binary internal relations by the study of jointly monic spans of which one leg is surjective, which would require a different approach to the one followed here.

In the present work we have shown that a relative notion of normal monomorphism can be defined as the zero-class of a certain type of internal binary relations. Every subcategory $\mathcal{R}$ of internal binary relations on a pointed category $\mathbf{C}$, for which the normalization functor admits a left adjoint, gives rise to a corresponding class of normal monomorphisms as the zero classes of the relations in $\mathcal{R}$. We have studied the particular case where $\mathbf{C} = \mathbf{Mon}$ and $\mathcal{R}$ consists of reflexive relations, preorders, and congruences, from which the respective zero-classes are clots, positive cones, and classical normal subobjects. It is clear that other situations may be of interest too, namely reflexive and symmetric relations.

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