

Gabor Frames in Quaternionic Analysis: A Transition from Euclidean to Hyperbolic Structures¹

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Abstract

In this paper, we introduce a novel integral transform termed the two-sided Quaternion Hyperbolic Windowed Fourier Transform (ts-QHWFT), designed for the analysis of two-dimensional quaternion-valued signals defined in an open rectangle of the plane equipped with a hyperbolic measure. We investigate the basic properties of the ts-QHWFT, including modulation, hyperbolic translation, hyperbolic dilation, Moyal's identity, and an inversion formula, which are essential for our analysis. The proposed ts-QHWFT enables the construction of quaternionic hyperbolic Gabor frames employing innovative time-frequency analysis tools such as the relativistic version of the Wiener space of quaternion-valued signals, the hyperbolic Poisson summation formula, and the correlation function. We then establish hyperbolic analogs of Walnut's and Janssen's representations of the quaternionic Gabor frame, as well as quaternionic versions of the Wexler-Raz biorthogonality and Ron-Shen duality. We also provide a detailed characterization of quaternionic hyperbolic tight Gabor frames. This algebraic approach provides a hyperbolic counterpart of the corresponding results in the quaternionic Euclidean scenario. Finally, as an application, we demonstrate the utility of the proposed ts-QHWFT for the analysis of two-dimensional linear space-varying hyperbolic systems, illustrating the results of this work.

Keywords: Quaternion hyperbolic windowed Fourier transform, Quaternionic hyperbolic Poisson summation formula, Quaternionic hyperbolic Gabor frames, Quaternion hyperbolic frame operator, Walnut's representation, Janssen's representation, Wexler-Raz biorthogonality.

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1. Introduction

Much research has focused on generalizing the real and complex Fourier Transforms (FTs) to higher dimensions. The four-dimensional Hamilton's quaternions, a natural extension of real and complex numbers, have garnered attention due to their multiple applications in applied mathematics, physics, and engineering. The generalization of the standard FT to quaternion space has led to various forms of the Quaternion Fourier Transform (QFT) with non-commutative exponential kernels, including the right-sided, the left-sided, and the two-sided QFTs. Most of the primary results for the QFTs have been established within Euclidean geometry, notably in the study of Payley-Wiener theorems [16, 20, 39] and uncertainty principles [3, 8, 26, 28, 37, 38, 44, 45]. Extensions of the results of the QFTs to the hyperbolic case, the so-called Quaternion Hyperbolic Fourier Transforms

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(QHFTs), were recently developed by Ferreira and Morais [18] for the right-sided and the two-sided QHFTs and extended for the right-sided Quaternion Hyperbolic Linear Canonical Transform in [40]. The standard FT lacks the capability to capture local information of non-stationary signals. The Windowed Fourier Transform (WFT), a generalization of the FT, has gained considerable attention due to its effectiveness in treating mathematical problems that the FT could not otherwise handle, with important applications in signal analysis, optics, and signal recovery. The main advantage of the WFT is the flexibility in the local spectral and temporal information received from a signal. The Heisenberg-Weyl uncertainty principle can easily demonstrate this circumstance, proven for the FT, the QFT [8], and recently for the QHFT [18]. We overcome this issue by multiplying the signal with translations and modulations of a given window function. This window plays a fundamental role in reconstructing the (modified) signal. The Quaternion Windowed Fourier Transform (QWFT) was introduced in [4] (see also [5, 19, 31]). These results were later refined by Ell et al. [15], who developed an operational calculus for this transform, further highlighting its significance in the context of hypercomplex signal analysis and processing.

In [4, 5], Bahri et al. conducted an in-depth study of the right-sided QWFT, establishing its fundamental properties, such as the reconstruction formula, orthogonality relations, isometry property, and a Heisenberg-type uncertainty principle for this transform. However, the non-commutativity of quaternions presents significant challenges when attempting to extend properties like shift and modulation using this approach. In [19], the authors defined the two-sided QWFT and proved the Balian-Low Theorem for a quaternionic Gabor system associated with discrete versions of the kernels of the right-sided QWFT. This system forms an orthonormal basis for the Hilbert space of square-integrable quaternion-valued Lebesgue measurable functions. The two-sided version of the QWFT has played a primary role in understanding and solving problems in signal analysis and processing, and it is linked to Bülow's notion of a hypercomplex signal [6]. Further developments on the QWFT and its relation to quaternionic Gabor frames can be found in [32–36].

While the existing results for the right-sided and two-sided QWFTs are captivating in the continuous case, from a practical standpoint, it becomes necessary to work with a discrete set of data. This data is typically structured to resemble a (multi-dimensional) lattice with parameters $\alpha, \beta > 0$. This setup led to the development of Gabor frames and the Gabor frame operator based on the two-sided QWFT using time-frequency analysis tools [7, 23–25]. The variety of these outcomes in signal processing underscores the need for a suitable hyperbolic analogue within the context of quaternionic analysis.

The goals of this article are as follows:

1. To introduce the two-sided Quaternion Hyperbolic Windowed Fourier Transform (ts-QHWFT) associated with quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure.
2. To derive the main properties of the new transform, including hyperbolic translation, modulation, hyperbolic dilation, Moyal's identity, and its inversion formula.
3. To describe the relativistic version of the Wiener space that arises from the periodization of quaternion-valued signals, along with the hyperbolic Poisson summation formula and the correlation function, and to compute the corresponding Gabor frames in a quaternionic Hilbert space.
4. To prove Walnut's and Janssen's representations of quaternionic Gabor frames and explore hyperbolic versions of duality principles, such as the Wexler-Raz biorthogonality relations and the Ron-Shen duality principle.
5. To apply the proposed ts-QHWFT to analyze two-dimensional linear space-varying hyperbolic systems.

The body of the present paper will cover the following topics: Section 2 introduces the notation

and preliminary material, focusing on the necessary results required for the subsequent sections. Section 3 presents the ts-QHWFT using the hyperbolic geometry language and derives some of its elementary properties, such as hyperbolic translation, modulation, hyperbolic dilation, Moyal's identity, and the inversion formula. The ts-QHWFT is defined by replacing the Euclidean plane waves with corresponding hyperbolic relativistic plane waves of the FT in one dimension, where two quaternion algebra generators replace the role of the imaginary unit, a fundamental feature to maintain the separation between the two dimensions. The non-commutativity of the Hamiltonian quaternion algebra allows for different forms of the QHWFT. In Section 4, we use the proposed ts-QHWFT to construct quaternionic hyperbolic Gabor frames, employing time-frequency analysis tools such as the relativistic version of the Wiener space of quaternion-valued signals and the hyperbolic Poisson summation formula. Besides, we prove hyperbolic analogs of Walnut's and Janssen's representations of quaternionic Gabor frames, as well as quaternionic versions of the Wexler-Raz biorthogonality relations and the Ron-Shen duality principle. Besides, a detailed characterization of quaternionic hyperbolic tight Gabor frames associated with the new transform is also investigated. In Section 5, we illustrate the application of the ts-QHWFT by analyzing two-dimensional linear space-varying hyperbolic systems. Concluding notes are given in Section 6.

The proposed ts-QHWFT not only extends the known properties of the ts-QWFT in the standard Euclidean case but also unifies the rich canonical structure of Quaternionic Analysis across both Euclidean and hyperbolic spaces. This unifying aspect underscores the novelty of our research within Quaternionic Analysis, making it a significant and compelling area of further study.

2. Preliminaries and terminology

2.1. Hyperbolic tools

Recent literature has highlighted the use of hyperbolic manifolds as embedding spaces for machine learning and computational tasks (e.g., [2] and references therein). This growing interest underscores the importance of developing tools for hyperbolic signal analysis and processing, a key aspect of our research.

For the extension of Euclidean geometry to the hyperbolic case, we first need to set the background based on a relativistic setting. Let $t_1, t_2 > 0$ be any fixed positive constants. We consider an open rectangle centered at the origin in the two-dimensional Euclidean space $\mathbb{R}^2$ defined by

$$\begin{aligned}\mathbb{R}_{t_1, t_2}^2 &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < t_1, |x_2| < t_2, t_1, t_2 \in \mathbb{R}^+\} \\ &= \mathbb{R}_{t_1} \times \mathbb{R}_{t_2},\end{aligned}\tag{1}$$

where $\mathbb{R}_{t_k} = (-t_k, t_k)$, $k = 1, 2$.

The relativistic (hyperbolic) addition and the scalar multiplication in $\mathbb{R}_{t_1, t_2}^2$ are defined componentwise by

$$\mathbf{x} \oplus \mathbf{y} = (x_1 \oplus y_1, x_2 \oplus y_2) = \left(\frac{x_1 + y_1}{1 + \frac{x_1 y_1}{t_1^2}}, \frac{x_2 + y_2}{1 + \frac{x_2 y_2}{t_2^2}} \right)\tag{2}$$

and

$$\lambda \otimes \mathbf{x} = (\lambda \otimes x_1, \lambda \otimes x_2) = \left(t_1 \tanh\left(\lambda \tanh^{-1}\left(\frac{x_1}{t_1}\right)\right), t_2 \tanh\left(\lambda \tanh^{-1}\left(\frac{x_2}{t_2}\right)\right) \right)\tag{3}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{t_1, t_2}^2$ and all $\lambda \in \mathbb{R}$.

It turns out that the abelian group $(\mathbb{R}_{t_1, t_2}^2, \oplus)$ is a vector space $(\mathbb{R}_{t_1, t_2}^2, \oplus, \otimes)$. In this notation, the additive identity for the vector space is the element $\mathbf{0} = (0, 0) \in \mathbb{R}_{t_1, t_2}^2$, satisfying $\mathbf{0} \oplus \mathbf{x} = \mathbf{x}$ for all

$\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$, and the additive inverse for the element $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$ is $\ominus \mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$, such that $\ominus \mathbf{x} \oplus \mathbf{x} = \mathbf{0}$. (We note that in this case, $\ominus \mathbf{x} = -\mathbf{x}$.) Moreover, it follows that $\mathbf{x} \ominus \mathbf{y} = \mathbf{x} \oplus (\ominus \mathbf{y})$. It is to be noted that in the limit of large t_1 and t_2 , $t_1, t_2 \rightarrow \infty$, the rectangle $\mathbb{R}_{t_1, t_2}^2$ expands to the whole of the space $\mathbb{R}^2$. As we see from (2) and (3), $\lim_{t_1, t_2 \rightarrow \infty} \mathbf{x} \oplus \mathbf{y} = (x_1 + y_1, x_2 + y_2)$ and $\lim_{t_1, t_2 \rightarrow \infty} \lambda \otimes \mathbf{x} = (\lambda x_1, \lambda x_2)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{t_1, t_2}^2$ and all $\lambda \in \mathbb{R}$. These limits will play a fundamental role in subsequent sections.

The isomorphism between $(\mathbb{R}_{t_1, t_2}^2, \oplus, \otimes)$ and $(\mathbb{R}^2, +, \times)$ is manifested by the mapping

$$h(\mathbf{x}) = \left(t_1 \tanh^{-1} \left(\frac{x_1}{t_1} \right), t_2 \tanh^{-1} \left(\frac{x_2}{t_2} \right) \right), \quad \mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$$

and both algebraic structures agree in the limit of large t_1 and t_2 , $t_1, t_2 \rightarrow \infty$.

In what follows, we found it convenient to introduce a special symbol to denote the extension of the hyperbolic space variable $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$ to the whole of the space $\mathbb{R}^2$ by setting

$$\underline{\mathbf{x}} = (\underline{x}_1, \underline{x}_2) = \left(t_1 \tanh^{-1} \left(\frac{x_1}{t_1} \right), t_2 \tanh^{-1} \left(\frac{x_2}{t_2} \right) \right) \in \mathbb{R}^2.$$

Moreover, the restriction of the variable $\mathbf{x} \in \mathbb{R}^2$ to $\mathbb{R}_{t_1, t_2}^2$ will be denoted by $\tilde{\mathbf{x}}$ by setting

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = \left(t_1 \tanh \left(\frac{x_1}{t_1} \right), t_2 \tanh \left(\frac{x_2}{t_2} \right) \right) \in \mathbb{R}_{t_1, t_2}^2. \quad (4)$$

2.2. Quaternions, quaternion modules, and the two-sided QHFT

Hamilton introduced the *quaternion algebra* $\mathbb{H}$ as an extension of the complex numbers to four dimensions. This algebra can be described as the set of elements of the form

$$q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k},$$

where the q_i 's ($i = 0, 1, 2, 3$) are the components of the quaternion q which are assumed to be real, and the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the elements of the basis of the algebra satisfying besides the associative law also the multiplication rules given below

$$\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

A quaternion q can be written as $q = q_0 + \underline{q}$, where the *scalar* and *vector parts* of q are defined, respectively, by $\text{Sc}(q) = q_0$ and $\underline{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. The *conjugate* of a quaternion q is defined by

$$\bar{q} = \text{Sc}(q) - \underline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3,$$

with the property that the conjugate of a product is the product of the conjugates in the reverse order, i.e., $\overline{pq} = \bar{q}\bar{p}$, for all $p, q \in \mathbb{H}$. The (*algebraic*) *norm* of q is defined by $|q|^2 = q\bar{q} = \bar{q}q = \sum_{i=0}^3 q_i^2$. It further follows that $|pq| = |p||q|$, for all $p, q \in \mathbb{H}$.

In this paper, we are concerned with the class of quaternionic signals

$$f: \mathbb{R}_{t_1, t_2}^2 \rightarrow \mathbb{H}$$

such that for $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$,

$$f(\mathbf{x}) = f_0(\mathbf{x}) + \mathbf{i}f_1(\mathbf{x}) + \mathbf{j}f_2(\mathbf{x}) + \mathbf{k}f_3(\mathbf{x})$$

or, equivalently, in a symmetric form as

$$f(\mathbf{x}) = f_0(\mathbf{x}) + \mathbf{i}f_1(\mathbf{x}) + f_2(\mathbf{x})\mathbf{j} + \mathbf{i}f_3(\mathbf{x})\mathbf{j}, \quad (5)$$

where the f_i 's ($i = 0, 1, 2, 3$) are real-valued functions defined in $\mathbb{R}_{t_1, t_2}^2$.

Definition 2.1. A function $f: \mathbb{R}_{t_1, t_2}^2 \rightarrow \mathbb{H}$ is said to be even in the first variable if $f(-x_1, x_2) = f(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}_{t_1, t_2}^2$.

Given the tensorial nature of the quaternionic kernels appearing in the two-dimensional QHFTs, it is convenient to introduce the *quaternionic hyperbolic relativistic plane waves* on $\mathbb{R}_{t_k}$, $k = 1, 2$.

Definition 2.2 ([18]). Let $t_1, t_2 \in \mathbb{R}^+$, and let $q = \mathbf{i}$ for $k = 1$ and $q = \mathbf{j}$ for $k = 2$. For $(x_1, x_2) \in \mathbb{R}_{t_1, t_2}^2$ and $(\omega_1, \omega_2) \in \mathbb{R}^2$, the 1D quaternionic hyperbolic plane waves are defined by

$$e_{\omega_k, t_k}(x_k) = \left(\frac{1 + \frac{x_k}{t_k}}{1 - \frac{x_k}{t_k}} \right)^{-\pi q \omega_k t_k}, \quad k = 1, 2. \quad (6)$$

It turns out that we can write (6) as

$$e_{\omega_k, t_k}(x_k) = e^{-\pi q \omega_k t_k \ln \left(\frac{1 + \frac{x_k}{t_k}}{1 - \frac{x_k}{t_k}} \right)} = e^{-2\pi q \omega_k t_k \tanh^{-1} \left(\frac{x_k}{t_k} \right)} = e^{-2\pi q \omega_k x_k}. \quad (7)$$

Here $q = \mathbf{i}$ for $k = 1$ and $q = \mathbf{j}$ for $k = 2$. (We will often adopt this notation throughout the paper.)

Since the quaternionic multiplication is generally not commutative,

$$e^{-2\pi(p+r)x_k \omega_k} = e^{-2\pi p x_k \omega_k} e^{-2\pi r x_k \omega_k}$$

does not hold unless the quaternions p and r commute.

Definition 2.3. The spaces $L^p(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, $1 \leq p < \infty$, are defined to be the left modules of all measurable quaternion-valued functions f defined on $\mathbb{R}_{t_1, t_2}^2$ satisfying the condition $|f|^p \in L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{R})$ (abbreviated as $L^1(\mathbb{R}_{t_1, t_2}^2)$), with norm

$$\|f\|_{L^p(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = \left(\int_{\mathbb{R}_{t_1, t_2}^2} |f(\mathbf{x})|^p d\mu(\mathbf{x}) \right)^{1/p} < \infty,$$

where $d\mu(\mathbf{x})$ denotes the hyperbolic measure on $\mathbb{R}_{t_1, t_2}^2$ given by

$$d\mu(\mathbf{x}) = d\mu(x_1) d\mu(x_2) = \left(1 - \frac{x_1^2}{t_1^2}\right)^{-1} \left(1 - \frac{x_2^2}{t_2^2}\right)^{-1} dx_1 dx_2.$$

For $p = \infty$, the space $L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ contains the essentially bounded measurable functions $f: \mathbb{R}_{t_1, t_2}^2 \rightarrow \mathbb{H}$ with norm

$$\|f\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = \text{ess sup}_{\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2} |f(\mathbf{x})| < \infty.$$

The notation used above conceals the fact that $d\mu(\mathbf{x})$ depends on t_1 and t_2 .

The primary space $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ is endowed with the usual definition of the (left-) *quaternionic inner product* given by

$$(f, g)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = \int_{\mathbb{R}_{t_1, t_2}^2} f(\mathbf{x}) \overline{g(\mathbf{x})} d\mu(\mathbf{x}) \quad (8)$$

for all $f, g \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. It is a (left) quaternionic Hilbert space with the associated norm $\|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = (f, f)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^{1/2}$.

We can also endow the space $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ with a positive definite symmetric bilinear form (known as the *scalar inner product*), defined by

$$\langle f, g \rangle_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = \text{Sc}((f, g)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}) = \int_{\mathbb{R}_{t_1, t_2}^2} \text{Sc}(f(\mathbf{x})\overline{g(\mathbf{x})}) d\mu(\mathbf{x}). \quad (9)$$

It turns out that (9) does not define an inner product in $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ seen as an $\mathbb{H}$ -linear space because it is not homogeneous for quaternionic scalars.

Definition 2.4 ([18]). *We say that an $\mathbb{H}$ -valued function f on $\mathbb{R}_{t_1, t_2}^2$ is rapidly decreasing if for every $m \geq 0$, $\sup_{\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2} |\underline{\mathbf{x}}|^m |f(\mathbf{x})| < \infty$. We further denote by $\mathcal{S}(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ the Schwartz space of all rapidly decreasing $\mathbb{H}$ -valued smooth functions on $\mathbb{R}_{t_1, t_2}^2$, defined by*

$$\mathcal{S}(\mathbb{R}_{t_1, t_2}^2, \mathbb{H}) = \left\{ f \in C^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H}) : \forall \alpha, \beta \in \mathbb{N}^2, \sup_{\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2} |\underline{\mathbf{x}}^\alpha \partial_h^\beta f(\mathbf{x})| < \infty \right\},$$

where $\underline{\mathbf{x}}^\alpha = \underline{x}_1^{\alpha_1} \underline{x}_2^{\alpha_2}$. The hyperbolic partial derivatives $\partial_h f / \partial x_k$, $k = 1, 2$, are defined by

$$\frac{\partial_h f}{\partial x_1}(x_1, x_2) = \lim_{\epsilon \rightarrow 0} \frac{f(x_1 \oplus \epsilon, x_2) - f(x_1, x_2)}{\epsilon}$$

and

$$\frac{\partial_h f}{\partial x_2}(x_1, x_2) = \lim_{\epsilon \rightarrow 0} \frac{f(x_1, x_2 \oplus \epsilon) - f(x_1, x_2)}{\epsilon}.$$

We have the following definition of the two-sided QHFT, which gives the hyperbolic counterpart of the corresponding Euclidean two-sided QFT [13–15] (see also [9, 27, 29]).

Definition 2.5 ([18]). *The two-sided Quaternionic Hyperbolic Fourier Transform (ts-QHFT) of a quaternionic signal f is defined as $\mathcal{F}_{QH}(f): \mathbb{R}_{t_1, t_2}^2 \rightarrow \mathbb{H}$, where*

$$\boldsymbol{\omega} = (\omega_1, \omega_2) \mapsto \mathcal{F}_{QH}(f)(\boldsymbol{\omega}) = \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi \mathbf{i} \underline{x}_1 \omega_1} f(\mathbf{x}) e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2} d\mu(\mathbf{x}). \quad (10)$$

A sufficient condition for the integral to exist is that $f \in L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. It can also be shown that the integral exists if $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$.

We refer to $\mathbf{x}$ as the hyperbolic-space variable and $\boldsymbol{\omega}$ as the angular-frequency variable.

Since $|e^{-2\pi \mathbf{i} \underline{x}_1 \omega_1} f(\mathbf{x}) e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2}| = |f(\mathbf{x})|$ for all $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$ and $(\omega_1, \omega_2) \in \mathbb{R}^2$, it follows that if f is absolutely integrable in $\mathbb{R}_{t_1, t_2}^2$, then the ts-QHFT given as (10) is well-defined, and the corresponding integral converges absolutely for $(\omega_1, \omega_2) \in \mathbb{R}^2$. Additionally, it is important to maintain a fixed order for the factors in (10) because the kernels $e^{-2\pi q \underline{x}_k \omega_k}$, $k = 1, 2$, do not generally commute with every element of the quaternion algebra.

Using the mapping (4), we may expand the proposed ts-QHFT to the whole of the space $\mathbb{R}^2$ as follows:

$$\mathcal{F}_{QH}(f)(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-2\pi \mathbf{i} x_1 \omega_1} f(\tilde{\mathbf{x}}) e^{-2\pi \mathbf{j} x_2 \omega_2} dx_1 dx_2.$$

By (5), it follows that $\mathcal{F}_{QH}(f) = \hat{f}$ has the symmetric representation into $\mathbb{H}$ -valued functions:

$$\hat{f}(\boldsymbol{\omega}) = \hat{f}_0(\boldsymbol{\omega}) + \mathbf{i} \hat{f}_1(\boldsymbol{\omega}) + \hat{f}_2(\boldsymbol{\omega}) \mathbf{j} + \mathbf{i} \hat{f}_3(\boldsymbol{\omega}) \mathbf{j},$$

where $\hat{f}_i = \mathcal{F}_{QH}(f_i)(\omega)$ for $i = 0, 1, 2, 3$.

We refer to [18] for detailed information on the fundamental properties of the ts-QHFT, including linearity, symmetry, modulation, hyperbolic translation, and scaling. In addition, the inversion formula, the Plancherel and Parseval Theorems for the ts-QHFT were also derived in [18]. These results are as follows:

Theorem 2.1 (Inversion formula). *Let $f \in L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. The components of f can be reconstructed via the inversion formula*

$$f_i(\mathbf{x}) = \mathcal{F}_{QH}^{-1}(\hat{f}_i)(\mathbf{x}) = \int_{\mathbb{R}^2} e^{2\pi i \underline{x}_1 \omega_1} \hat{f}_i(\omega) e^{2\pi j \underline{x}_2 \omega_2} d\omega, \quad i = 0, 1, 2, 3 \quad (11)$$

for a.e. $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$, whenever the integral exists. A sufficient condition for the integral to exist is that $\mathcal{F}_{QH}(f) \in L^1(\mathbb{R}^2, \mathbb{H})$.

Theorem 2.2 (Plancherel's Theorem). *Let $f, g \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. The two-sided QHFT (10) satisfies Plancherel's identity with respect to the scalar product (9), namely*

$$\langle \mathcal{F}_{QH}(f), \mathcal{F}_{QH}(g) \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle f, g \rangle_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \quad (12)$$

Further, if f (or g) is an even function in the first variable then Plancherel's identity also holds for the quaternionic product (8), i.e.,

$$(\mathcal{F}_{QH}(f), \mathcal{F}_{QH}(g))_{L^2(\mathbb{R}^2, \mathbb{H})} = (f, g)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \quad (13)$$

Corollary 2.1 (Parseval's Theorem). *Let $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. The two-sided QHFT (10) satisfies Parseval's identity, namely*

$$\|\mathcal{F}_{QH}(f)\|_{L^2(\mathbb{R}^2, \mathbb{H})} = \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \quad (14)$$

In Section 3, we take a step further and introduce the ts-QHWFT, a novel extension of the ts-QHFT that leads to the hyperbolic analogue of the Euclidean ts-QWFT.

2.3. Frame theory revisited

The theory of frames, first introduced by R.J. Duffin and A.C. Schaeffer [11], has been widely studied. Frames are applied in diverse areas, including sampling theory [1], operator theory [22], and wireless communications [42]. They offer a generalization of the concept of a basis in separable Hilbert spaces.

Definition 2.6 ([10]). *A sequence $\{e_n : n \in \mathbb{N}\}$ in a (separable) Hilbert space $\mathcal{H}$ is called a frame if there exist positive constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$,*

$$A \|f\|_{\mathcal{H}}^2 \leq \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle| \leq B \|f\|_{\mathcal{H}}^2. \quad (15)$$

The constants A and B in (15) are called frame bounds. We refer to a tight frame if $A = B$ and a Parseval frame if $A = B = 1$.

Definition 2.7 ([21]). *For any frame $\{e_n : n \in \mathbb{N}\}$, the analysis operator C is defined by*

$$Cf = \{\langle f, e_n \rangle : n \in \mathbb{N}\},$$

the reconstruction operator D for a finite sequence $c = \{c_n: n \in \mathbb{N}\}$ is given by

$$Dc = \sum_{n \in \mathbb{N}} c_n e_n,$$

and the frame operator S is then defined by

$$Sf = DCf = \sum_{n \in \mathbb{N}} \langle f, e_n \rangle e_n.$$

It is easy to see that the frame operator S is positive, self-adjoint, and invertible in $\mathcal{H}$.

Definition 2.8 ([21]). Let $\{e_n: n \in \mathbb{N}\}$ be a countable set in a Banach space $\mathcal{B}$. The series $\sum_{n \in \mathbb{N}} e_n$ is said to converge unconditionally to $f \in \mathcal{B}$, if for every $\epsilon > 0$, there exists a finite subset $F_0 \subseteq \mathbb{N}$ such that

$$\|f - \sum_{n \in F} e_n\|_{\mathcal{B}} < \epsilon$$

for all finite subsets $F \supseteq F_0$.

The following result characterizes expansions in a Hilbert space using the frame operator.

Theorem 2.3 ([21]). Let $\{e_n: n \in \mathbb{N}\}$ be a frame with frame operator S . Then every f in the Hilbert space $\mathcal{H}$ has the non-orthogonal expansion

$$f = \sum_{n \in \mathbb{N}} \langle f, S^{-1}e_n \rangle e_n, \quad \forall f \in \mathcal{H}.$$

The family $\{S^{-1}e_n: n \in \mathbb{N}\}$ is itself a frame, called the *canonical dual frame*. If $\{e_n: n \in \mathbb{N}\}$ is a frame but not a Riesz basis, there exists a *dual frame* $\{g_n: n \in \mathbb{N}\}$ such that

$$f = \sum_{n \in \mathbb{N}} \langle f, g_n \rangle e_n, \quad \forall f \in \mathcal{H}.$$

For a tight frame, we have $\langle Sf, f \rangle = A\|f\|_{\mathcal{H}}^2$, which implies that $S = AI$; whence,

$$f = \frac{1}{A} \sum_{n \in \mathbb{N}} \langle f, e_n \rangle e_n, \quad \forall f \in \mathcal{H}.$$

3. The ts-QHWFT

In this section, we introduce the ts-QHWFT and derive some of its properties, including Moyal's identity, its inversion formula, and the analogue of the Poisson summation formula. The ts-QHWFT is not linear with respect to quaternionic scalars. We further define the relativistic version of the Wiener space, which serves as an appropriate space for selecting window functions in hyperbolic Gabor analysis. We then prove a fundamental result about the boundedness of functions in the Wiener space. The treatment given here represents a generalization of the work in [7] and [24], underscoring the relevance of our contribution.

3.1. Definition and examples

We begin by introducing three classes of operators that will play a key role in our analysis of quaternionic Gabor frames. Their definitions are as follows. For $\mathbf{x}, \mathbf{b} \in \mathbb{R}_{t_1, t_2}^2$ and $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$,

$$T_{\mathbf{b}}f(\mathbf{x}) = f(\mathbf{x} \ominus \mathbf{b}), \quad \text{translation by } \mathbf{b}, \quad (16)$$

$$E(f) = \int_{\mathbb{R}_{t_1, t_2}^2} f(\mathbf{x}) d\mu(\mathbf{x}), \quad \text{mean operator}, \quad (17)$$

$$M_{\boldsymbol{\omega}}(f)(\mathbf{x}) = e^{-2\pi\mathbf{i}x_1\omega_1} f(\mathbf{x}) e^{-2\pi\mathbf{j}x_2\omega_2}, \quad \text{modulation by } \boldsymbol{\omega}. \quad (18)$$

For $\mathbf{c} = (c_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in \ell^2(\mathbb{H})$ (i.e., quaternionic sequences $(c_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2}$ such that $\sum_{n_1, n_2=1}^{\infty} |c_{\mathbf{n}}|^2 < \infty$), we consider the modulation by $\mathbf{x}$ acting on $\mathbf{c}$ by

$$\mathbf{c} \mapsto M'_{\mathbf{x}; \mathbf{n}}(\mathbf{c}) = (e^{-2\pi\mathbf{i}x_1n_1} c_{\mathbf{n}} e^{-2\pi\mathbf{j}x_2n_2})_{\mathbf{n} \in \mathbb{Z}^2}. \quad (19)$$

We note that the operators $M_{\boldsymbol{\omega}}$ and $M'_{\mathbf{x}; \mathbf{n}}$ are quaternionic non-linear operators. Further, these operators can be used to describe the ts-QHFT (10) as follows:

$$\mathcal{F}_{QH}(f)(\boldsymbol{\omega}) = E(M_{\boldsymbol{\omega}}(f)).$$

Now, we are in a position to introduce the ts-QHWFT, which gives the hyperbolic counterpart of the corresponding Euclidean ts-QWFT introduced by Bahri et al. in [4]. Since the ts-QHFT only gives global information about the behavior of f , we use a real-valued window function to obtain information on the local properties of the signal. (The derivation of the ts-QHWFT with respect to a quaternion-valued window function is also possible, but we do not dwell further here on this structure.)

Definition 3.1 (The ts-QHWFT). *Let $g \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{R})$ be a non-zero, real-valued function. The ts-QHWFT of $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ with respect to the window (or cut-off) g is defined as*

$$\begin{aligned} \mathcal{Q}_g f(\boldsymbol{\omega}, \mathbf{b}) &= \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi\mathbf{i}x_1\omega_1} f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) e^{-2\pi\mathbf{j}x_2\omega_2} d\mu(\mathbf{x}) \\ &= E(M_{\boldsymbol{\omega}}(f T_{\mathbf{b}} g)), \end{aligned} \quad (20)$$

where $(\boldsymbol{\omega}, \mathbf{b}) \in \mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2$.

The proposed ts-QHWFT may be expanded to the whole of the space $\mathbb{R}^2$ as follows:

$$\mathcal{Q}_g f(\boldsymbol{\omega}, \mathbf{b}) = \int_{\mathbb{R}^2} e^{-2\pi\mathbf{i}x_1\omega_1} f(\tilde{x}_1, \tilde{x}_2) g(\tilde{x}_1 - \tilde{b}_1, \tilde{x}_2 - \tilde{b}_2) e^{-2\pi\mathbf{j}x_2\omega_2} dx_1 dx_2.$$

Some examples of possible two-dimensional window functions are the *hyperbolic Gaussian function* defined by

$$g_1(\mathbf{x}; \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x_1^2 + x_2^2}{2\sigma^2}}, \quad \sigma > 0, \quad (21)$$

and the *hyperbolic Haar window function* defined by

$$\begin{aligned} g_2(\mathbf{x}) &= \begin{cases} 1, & 0 \leq x_1/t_1 \leq \tanh\left(\frac{1}{2t_1}\right), 0 \leq x_2/t_2 \leq \tanh\left(\frac{1}{2t_2}\right) \\ -1, & \tanh\left(\frac{1}{2t_1}\right) < x_1/t_1 \leq \tanh\left(\frac{1}{t_1}\right), \tanh\left(\frac{1}{2t_2}\right) < x_2/t_2 \leq \tanh\left(\frac{1}{t_2}\right) \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (22)$$

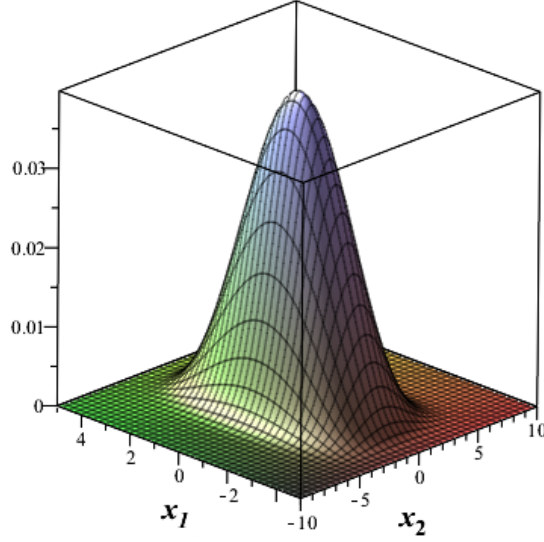


Figure 1: Representation of the function (21) with $\sigma = 2$ in $\mathbb{R}_{5,10}^2$.

Figure 1 shows the hyperbolic Gaussian function in the hyperbolic spatial domain $\mathbb{R}_{5,10}^2$.

We give a first example, calculating the ts-QHWFT of the hyperbolic Gaussian function $f(\mathbf{x}) = g_1(\mathbf{x}; \sigma)$ using the Haar window function (22). From Definition 3.1, we have

$$\begin{aligned} \mathcal{Q}_{g_2}f(\omega, \mathbf{b}) &= \frac{1}{2\pi\sigma^2} \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi i \underline{x}_1 \omega_1} e^{-(\underline{x}_1^2 + \underline{x}_2^2)/(2\sigma^2)} g_2(\mathbf{x} \ominus \mathbf{b}) e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2} d\mu(\mathbf{x}) \\ &= \frac{1}{2\pi\sigma^2} \int_{b_1}^{\tilde{\frac{1}{2}} \oplus b_1} e^{-2\pi i \underline{x}_1 \omega_1} e^{-\underline{x}_1^2/(2\sigma^2)} d\mu(x_1) \int_{b_2}^{\tilde{\frac{1}{2}} \oplus b_2} e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2} e^{-\underline{x}_2^2/(2\sigma^2)} d\mu(x_2) \\ &\quad - \frac{1}{2\pi\sigma^2} \int_{\tilde{\frac{1}{2}} \oplus b_1}^{\tilde{1} \oplus b_1} e^{-2\pi i \underline{x}_1 \omega_1} e^{-\underline{x}_1^2/(2\sigma^2)} d\mu(x_1) \int_{\tilde{\frac{1}{2}} \oplus b_2}^{\tilde{1} \oplus b_2} e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2} e^{-\underline{x}_2^2/(2\sigma^2)} d\mu(x_2). \end{aligned}$$

After the change of variables $\underline{x}_k = y_k$, $k = 1, 2$, we obtain

$$\begin{aligned} \mathcal{Q}_{g_2}f(\omega, \mathbf{b}) &= \frac{1}{2\pi\sigma^2} \int_{b_1}^{\frac{1}{2} + b_1} e^{-2\pi i y_1 \omega_1} e^{-y_1^2/(2\sigma^2)} dy_1 \int_{b_2}^{\frac{1}{2} + b_2} e^{-2\pi \mathbf{j} y_2 \omega_2} e^{-y_2^2/(2\sigma^2)} dy_2 \\ &\quad - \frac{1}{2\pi\sigma^2} \int_{\frac{1}{2} + b_1}^{1 + b_1} e^{-2\pi i y_1 \omega_1} e^{-y_1^2/(2\sigma^2)} dy_1 \int_{\frac{1}{2} + b_2}^{\frac{1}{2} + b_2} e^{-2\pi \mathbf{j} y_2 \omega_2} e^{-y_2^2/(2\sigma^2)} dy_2. \end{aligned}$$

These integrals can be calculated using the erf function, $\text{erf}(z) = (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt$, $z \in \mathbb{C}$, resulting

in

$$\begin{aligned} \mathcal{Q}_{g_2} f(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{4\sigma^2} e^{-2\pi^2\sigma^2(\omega_1^2+\omega_2^2)} \\ &\left(\left(\operatorname{erf}\left(\frac{2i\pi\omega_1\sigma^2 + \underline{b}_1}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{4i\pi\omega_1\sigma^2 + 2\underline{b}_1 + 1}{2\sqrt{2}\sigma}\right) \right) \right. \\ &\quad \times \left(\operatorname{erf}\left(\frac{2j\pi\omega_2\sigma^2 + \underline{b}_2}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{4j\pi\omega_2\sigma^2 + 2\underline{b}_2 + 1}{2\sqrt{2}\sigma}\right) \right) \\ &\quad - \left(\operatorname{erf}\left(\frac{4i\pi\omega_1\sigma^2 + 2\underline{b}_1 + 1}{2\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{2i\pi\omega_1\sigma^2 + 2\underline{b}_1 + 1}{\sqrt{2}\sigma}\right) \right) \\ &\quad \times \left. \left(\operatorname{erf}\left(\frac{4j\pi\omega_2\sigma^2 + 2\underline{b}_2 + 1}{2\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{2j\pi\omega_2\sigma^2 + 2\underline{b}_2 + 1}{\sqrt{2}\sigma}\right) \right) \right). \end{aligned}$$

The four components of the ts-QHWFT separate four symmetry cases in the form

$$\begin{aligned} \mathcal{Q}_g f(\boldsymbol{\omega}, \mathbf{b}) &= \int_{\mathbb{R}_{t_1, t_2}^2} \cos(\underline{x}_1\omega_1) \cos(\underline{x}_2\omega_2) f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) d\mu(\mathbf{x}) \\ &\quad - i \int_{\mathbb{R}_{t_1, t_2}^2} \sin(\underline{x}_1\omega_1) \cos(\underline{x}_2\omega_2) f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) d\mu(\mathbf{x}) \\ &\quad - \int_{\mathbb{R}_{t_1, t_2}^2} \cos(\underline{x}_1\omega_1) \sin(\underline{x}_2\omega_2) f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) j d\mu(\mathbf{x}) \\ &\quad + i \int_{\mathbb{R}_{t_1, t_2}^2} \sin(\underline{x}_1\omega_1) \sin(\underline{x}_2\omega_2) f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) j d\mu(\mathbf{x}). \end{aligned} \quad (23)$$

Figure 2 shows some of the basis functions of the real part of the ts-QHWFT with respect to the window function $g_1(\mathbf{x}; 3)$ defined by (21) in the frequency domain $(\omega_1, \omega_2) \in \{1/16, 1/8, 1/4, 1/2, 1/220\} \times \{0, 1/2, 1/6, 1/12, 1/24\}$.

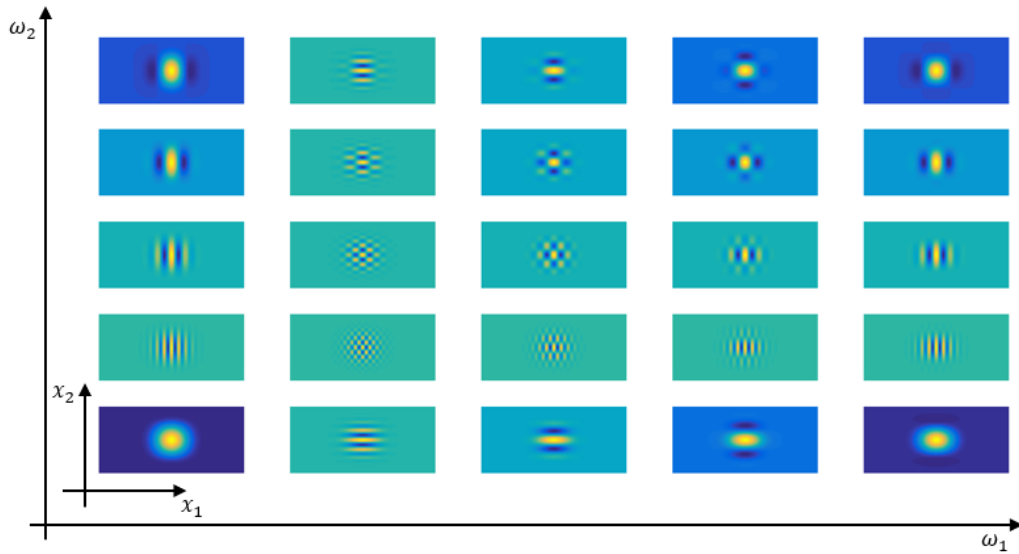


Figure 2: The small images are intensity images of the scalar part of the basis function in (23), for $t_1 = 20$ and $t_2 = 10$.

Now, employing (23), we show in Figure 3 the 4-components of the ts-QHWFT of the signal f

defined in $\mathbb{R}_{20,10}^2$ by

$$f(\mathbf{x}) = \cos(4x_1) e^{-x_1} \sin(2x_2) e^{-x_2} H(x_1) H(x_2),$$

where H is the *Heaviside function*, with respect to the *Haan window function* defined in $\mathbb{R}_{20,10}^2$ by

$$g(\mathbf{x}) = \cos^2(\pi x_1) \cos^2(\pi x_2), \quad -1/2 \leq x_1, x_2 \leq 1/2.$$

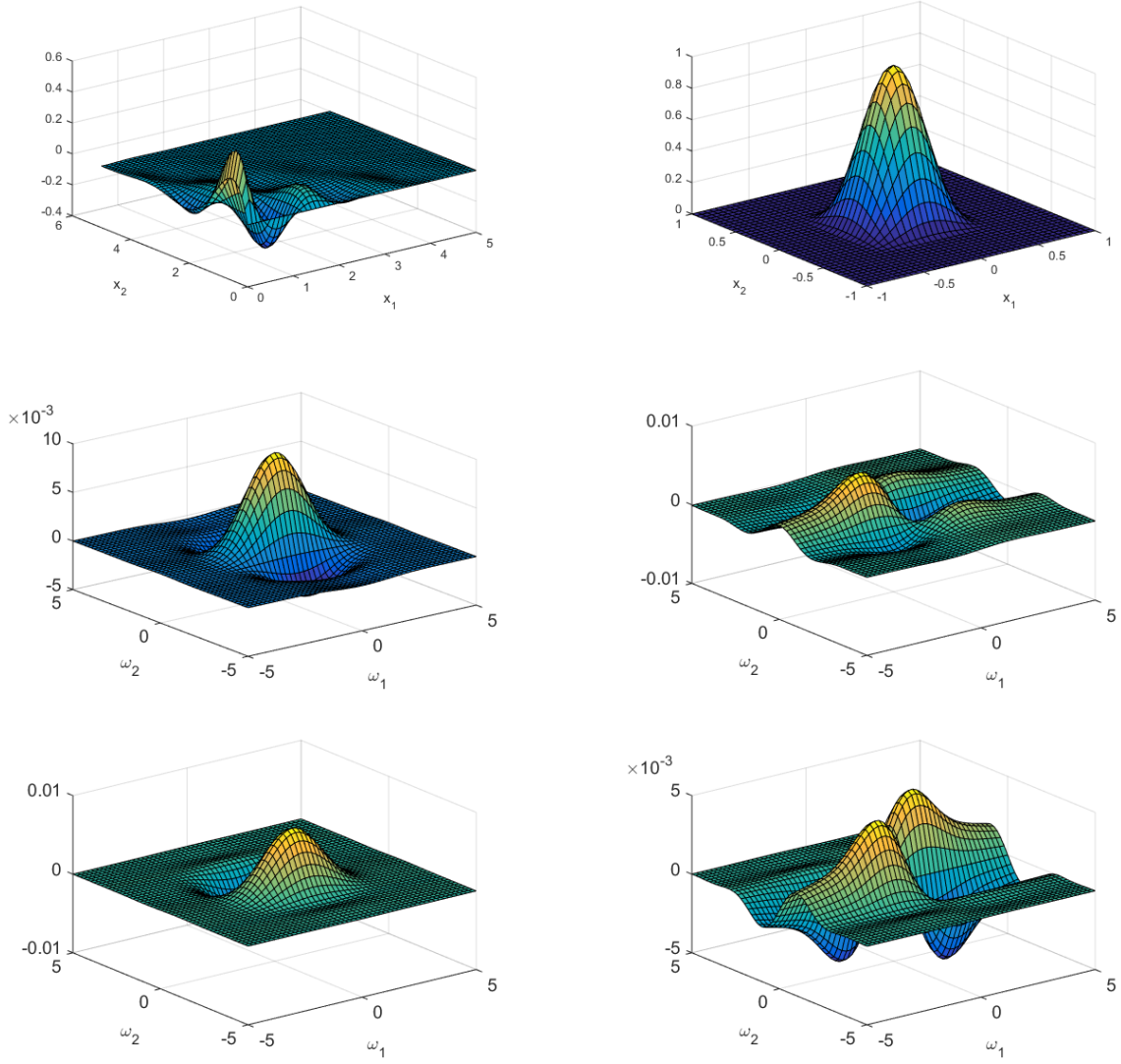


Figure 3: The signal f in the first quadrant of $\mathbb{R}_{20,10}^2$ (top left), the Haan window function with support in $[-1/2, 1/2]^2$ (top right), the real part of (23) (middle left), the $\mathbf{i}$ -component of (23) (middle right), the $\mathbf{j}$ -component of (23) (bottom left), and the $\mathbf{k}$ -component of (23) (bottom right).

3.2. Properties

In the following proposition, we state the main operational properties of the proposed ts-QHFFT. (We omit the proof of these properties since they are more or less analogous to the ones associated with the Euclidean ts-QHFT.)

Proposition 3.1. Let $f_1, f_2 \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, $g \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{R})$, $\mathbf{y} \in \mathbb{R}_{t_1, t_2}^2$, $\boldsymbol{\omega}_0 \in \mathbb{R}^2$, and $\lambda \in \mathbb{R} \setminus \{0\}$. We have

1. (Linearity)

- (a) $Q_g[\alpha f_1 + \beta f_2](\boldsymbol{\omega}, \mathbf{b}) = \alpha Q_g f_1(\boldsymbol{\omega}, \mathbf{b}) + \beta Q_g f_2(\boldsymbol{\omega}, \mathbf{b})$, for $\alpha, \beta \in \mathbb{R} \oplus \mathbf{i}\mathbb{R}$,
- (b) $Q_g[f_1 \alpha' + f_2 \beta'](\boldsymbol{\omega}, \mathbf{b}) = Q_g f_1(\boldsymbol{\omega}, \mathbf{b}) \alpha' + Q_g f_2(\boldsymbol{\omega}, \mathbf{b}) \beta'$, for $\alpha', \beta' \in \mathbb{R} \oplus \mathbf{j}\mathbb{R}$.

2. (Hyperbolic translation)

$$Q_g[T_{\mathbf{y}} f](\boldsymbol{\omega}, \mathbf{b}) = e^{-2\pi \mathbf{i} \underline{y}_1 \omega_1} Q_g f(\boldsymbol{\omega}, \mathbf{b} \ominus \mathbf{y}) e^{-2\pi \mathbf{j} \underline{y}_2 \omega_2}; \quad (24)$$

3. (Modulation)

$$Q_g[M_{-\boldsymbol{\omega}_0}(f)](\boldsymbol{\omega}, \mathbf{b}) = Q_g f(\boldsymbol{\omega} - \boldsymbol{\omega}_0, \mathbf{b}); \quad (25)$$

4. (Hyperbolic dilation/scaling)

$$Q_g[N_{\lambda_1, \lambda_2} f](\boldsymbol{\omega}, \mathbf{b}) = \frac{1}{|\lambda_1 \lambda_2|} Q_{(N_{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}} g)} f\left(\left(\frac{\omega_1}{\lambda_1}, \frac{\omega_2}{\lambda_2}\right), \lambda \otimes \mathbf{b}\right), \quad (26)$$

where $N_{\lambda_1, \lambda_2} f(\mathbf{x}) = f(\lambda_1 \otimes x_1, \lambda_2 \otimes x_2)$.

We will now derive two versions of Moyal's identity for the ts-QHWFT with respect to the quaternionic inner product (8) and the scalar inner product (9).

Theorem 3.1 (Moyal's identity). Let $f_1, f_2 \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ and $g_1, g_2 \in L^2(\mathbb{R}_{t_1, t_2}^2)$. Then $\mathcal{Q}_{g_k} f_k \in L^2(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ for $k = 1, 2$, and it satisfies Moyal's identity with respect to the scalar inner product (9), namely

$$\langle \mathcal{Q}_{g_1} f_1, \mathcal{Q}_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = (g_1, g_2)_{L^2(\mathbb{R}_{t_1, t_2}^2)} \langle f_1, f_2 \rangle_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \quad (27)$$

Further, if f_1 and g_1 (or f_2 and g_2) are even in the first variable, then Moyal's identity also holds with respect to the quaternionic inner product (8), i.e.,

$$(\mathcal{Q}_{g_1} f_1, \mathcal{Q}_{g_2} f_2)_{L^2(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = (g_1, g_2)_{L^2(\mathbb{R}_{t_1, t_2}^2)} (f_1, f_2)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \quad (28)$$

Proof. We first prove (28). Assume that the window functions g_1, g_2 are in $L^2(\mathbb{R}_{t_1, t_2}^2)$. As a result, $f_k T_{\mathbf{b}} g_k \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, $k = 1, 2$, for all $\mathbf{b} \in \mathbb{R}_{t_1, t_2}^2$. With the help of the proof of Plancherel's Theorem 2.2 for the ts-QHFT [18, Theorem 3.31] (which holds under the assumption of the even partial symmetry property in the first variable of f_1 and g_1 or f_2 and g_2) and Fubini's Theorem, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} \mathcal{Q}_{g_1} f_1(\boldsymbol{\omega}, \mathbf{b}) \overline{\mathcal{Q}_{g_2} f_2(\boldsymbol{\omega}, \mathbf{b})} d\boldsymbol{\omega} d\mu(\mathbf{b}) \\ &= \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} E(M_{\boldsymbol{\omega}}(f_1 T_{\mathbf{b}} g_1)) \overline{E(M_{\boldsymbol{\omega}}(f_2 T_{\mathbf{b}} g_2))} d\boldsymbol{\omega} d\mu(\mathbf{b}) \\ &= \int_{\mathbb{R}_{t_1, t_2}^2} \left(\int_{\mathbb{R}_{t_1, t_2}^2} f_1(\mathbf{x}) g_1(\mathbf{x} \ominus \mathbf{b}) \overline{f_2(\mathbf{x}) g_2(\mathbf{x} \ominus \mathbf{b})} d\mu(\mathbf{x}) \right) d\mu(\mathbf{b}) \\ &= \int_{\mathbb{R}_{t_1, t_2}^2} f_1(\mathbf{x}) \overline{f_2(\mathbf{x})} \left(\int_{\mathbb{R}_{t_1, t_2}^2} g_1(\mathbf{x} \ominus \mathbf{b}) \overline{g_2(\mathbf{x} \ominus \mathbf{b})} d\mu(\mathbf{b}) \right) d\mu(\mathbf{x}) \\ &= (f_1, f_2)(g_1, g_2). \end{aligned}$$

The last step follows from the change of variable $\mathbf{x} \ominus \mathbf{b} = \mathbf{y}$ and the hyperbolic translation invariance property of the measure $d\mu$. The proof of (27) is similar by employing (12). $\square$

Corollary 3.1. *If $f_1 = f_2 = f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ and $g_1 = g_2 = g \in L^2(\mathbb{R}_{t_1, t_2}^2)$, then we have*

$$\|\mathcal{Q}_g f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = \|g\|_{L^2(\mathbb{R}_{t_1, t_2}^2)} \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}.$$

As a consequence, for $g \in L^2(\mathbb{R}_{t_1, t_2}^2)$ with $\|g\|_{L^2(\mathbb{R}_{t_1, t_2}^2)} = 1$ we obtain Parseval's identity

$$\|\mathcal{Q}_g f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})},$$

for all $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, i.e., the proposed ts-QHWFT is an isometry from $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ to $L^2(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})$.

We now derive an inversion formula for the ts-QHWFT, which shows that the original signal f can be reconstructed from $\mathcal{Q}_{g_2}$ and g_1 by means of an inversion formula.

Theorem 3.2 (Inversion formula). *Let $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ and let $g_1, g_2 \in L^2(\mathbb{R}_{t_1, t_2}^2)$ be two non-zero, real-valued windows which are not orthogonal to each other over $\mathbb{R}_{t_1, t_2}^2$. The inversion formula for the ts-QHWFT given by (20) is*

$$f(\mathbf{x}) = \frac{1}{|(g_1, g_2)|} \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} e^{2\pi i \underline{x}_1 \omega_1} \mathcal{Q}_{g_2} f(\omega, \mathbf{b}) g_1(\mathbf{x} \ominus \mathbf{b}) e^{2\pi j \underline{x}_2 \omega_2} d\omega d\mu(\mathbf{b}) \quad (29)$$

whenever the integral exists. A sufficient condition for that is $\mathcal{F}_{QH}(\mathcal{Q}_{g_2} f) \in L^1(\mathbb{R}^2 \times \mathbb{R}_{t_1, t_2}^2, \mathbb{H})$.

Proof. A first observation shows that the ts-QHWFT (20) can be written as

$$\mathcal{Q}_{g_2} f(\omega, \mathbf{b}) = \mathcal{F}_{QH}(g_2(\mathbf{x} \ominus \mathbf{b}) f(\mathbf{x}))(\omega).$$

By applying the inversion (11) of the ts-QHFT, we obtain

$$\begin{aligned} g_2(\mathbf{x} \ominus \mathbf{b}) f(\mathbf{x}) &= \mathcal{F}_{QH}^{-1}[\mathcal{Q}_{g_2} f(\omega, \mathbf{b})](\mathbf{x}) \\ &= \int_{\mathbb{R}^2} e^{2\pi i \underline{x}_1 \omega_1} \mathcal{Q}_{g_2} f(\omega, \mathbf{b}) e^{2\pi j \underline{x}_2 \omega_2} d\omega. \end{aligned}$$

Multiplying both sides of the above relation by $g_1(\mathbf{x} \ominus \mathbf{b})$ and integrating over $\mathbb{R}_{t_1, t_2}^2$ with respect to $\mathbf{b}$, it follows that

$$\begin{aligned} f(\mathbf{x}) &\int_{\mathbb{R}_{t_1, t_2}^2} g_1(\mathbf{x} \ominus \mathbf{b}) g_2(\mathbf{x} \ominus \mathbf{b}) d\mu(\mathbf{b}) \\ &= \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} e^{2\pi i \underline{x}_1 \omega_1} \mathcal{Q}_{g_2} f(\omega, \mathbf{b}) g_1(\mathbf{x} \ominus \mathbf{b}) e^{2\pi j \underline{x}_2 \omega_2} d\omega d\mu(\mathbf{b}). \end{aligned}$$

The statement of the theorem follows from the relation

$$\int_{\mathbb{R}_{t_1, t_2}^2} g_1(\mathbf{x} \ominus \mathbf{b}) g_2(\mathbf{x} \ominus \mathbf{b}) d\mu(\mathbf{b}) = \int_{\mathbb{R}_{t_1, t_2}^2} g_1(\mathbf{b}) g_2(\mathbf{b}) d\mu(\mathbf{b}) = (g_1, g_2)_{L^2(\mathbb{R}_{t_1, t_2}^2)}.$$

□

The reconstruction formula (29) can be rewritten with the help of the operators defined above, which will be helpful for the rest of this paper. We have

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{|(g_1, g_2)|} \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} e^{2\pi i \underline{x}_1 \omega_1} \mathcal{Q}_{g_2} f(\omega, \mathbf{b}) e^{2\pi j \underline{x}_2 \omega_2} g_1(\mathbf{x} \ominus \mathbf{b}) d\omega d\mu(\mathbf{b}) \\ &= \frac{1}{|(g_1, g_2)|} \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} M_{-\omega} \mathcal{Q}_{g_2} f(\omega, \mathbf{b}) T_{\mathbf{b}} g_1(\mathbf{x}) d\omega d\mu(\mathbf{b}) \\ &= \frac{1}{|(g_1, g_2)|} \int_{\mathbb{R}_{t_1, t_2}^2} \int_{\mathbb{R}^2} M_{-\omega} E(M_{\omega}(f) T_{\mathbf{b}} g_1) T_{\mathbf{b}} g_1(\mathbf{x}) d\omega d\mu(\mathbf{b}). \end{aligned}$$

In the large limits of t_1 and t_2 , i.e., $t_1, t_2 \rightarrow \infty$, the inversion formula reduces to the corresponding result for the Euclidean ts-QWFT; see [7].

3.3. Quaternionic hyperbolic Gabor frames

The idea of periodization arises naturally in time-frequency analysis and Gabor frames [21]. We will now establish these ideas in the hyperbolic setting. From now on, we assume that the lattice point $\alpha = (\alpha_1, \alpha_2)$ is such that $0 < \alpha_1 < t_1$ and $0 < \alpha_2 < t_2$ for any fixed positive constants $t_1, t_2 > 0$, and denote by $Q_\alpha = [0, \alpha_1] \times [0, \alpha_2]$ a rectangle inside $\mathbb{R}_{t_1, t_2}^2$. Moreover, we denote by $Q = Q_{\tilde{\mathbf{1}}} = [0, t_1 \tanh(1/t_1)] \times [0, t_2 \tanh(1/t_2)]$ the hyperbolic unit rectangle inside $\mathbb{R}_{t_1, t_2}^2$, where $\tilde{\mathbf{1}} = (t_1 \tanh(1/t_1), t_2 \tanh(1/t_2))$.

We extend the fundamental property of periodization within the hyperbolic context as follows:

Lemma 3.1 (Periodization). *Let $f \in L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. For a given lattice point $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1 < t_1$ and $0 < \alpha_2 < t_2$, we have*

$$\int_{\mathbb{R}_{t_1, t_2}^2} f(\mathbf{x}) d\mu(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \int_{Q_\alpha} f(\mathbf{x} \oplus \mathbf{n} \otimes \alpha) d\mu(\mathbf{x}). \quad (30)$$

Proof. Since $f \in L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, by Fubini's Theorem, the integral and the sum on the right-hand side of (30) can be interchanged. Moreover, using the change of variable $\mathbf{x} \oplus \mathbf{n} \otimes \alpha = \mathbf{y}$ and the hyperbolic translation invariance property of the measure $d\mu$, direct computation shows that

$$\int_{Q_\alpha} \sum_{\mathbf{n} \in \mathbb{Z}^2} f(\mathbf{x} \oplus \mathbf{n} \otimes \alpha) d\mu(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \int_{Q_\alpha \oplus \mathbf{n} \otimes \alpha} f(\mathbf{y}) d\mu(\mathbf{y}) = \int_{\mathbb{R}_{t_1, t_2}^2} f(\mathbf{y}) d\mu(\mathbf{y}),$$

where the last equality holds since $\lim_{n_k \rightarrow \pm\infty} n_k \otimes \alpha_k = \pm t_k$, $k = 1, 2$, and hence the translated rectangles

$$Q_\alpha \oplus \mathbf{n} \otimes \alpha = [n_1 \otimes \alpha_1, (n_1 + 1) \otimes \alpha_1] \times [n_2 \otimes \alpha_2, (n_2 + 1) \otimes \alpha_2]$$

form a partition of $\mathbb{R}_{t_1, t_2}^2$. □

The following result extends Lemma 3.3, given in [18], which offers a quaternionic hyperbolic version of the Poisson summation formula associated with the proposed ts-QHWT.

Lemma 3.2 (Quaternionic hyperbolic Poisson summation formula). *Let $f \in \mathcal{S}(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. For a given lattice point $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1 < t_1$ and $0 < \alpha_2 < t_2$, the series*

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} f(\mathbf{x} \oplus \mathbf{n} \otimes \alpha)$$

converges in $L^1([0, \alpha_1] \times [0, \alpha_2])$, and

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} f(\mathbf{x} \oplus \mathbf{n} \otimes \alpha) = \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \sum_{\mathbf{m} \in \mathbb{Z}^2} e^{2\pi i \underline{x}_1 \frac{m_1}{\underline{\alpha}_1}} \mathcal{F}_{QH}(f) \left(\frac{m_1}{\underline{\alpha}_1}, \frac{m_2}{\underline{\alpha}_2} \right) e^{2\pi j \underline{x}_2 \frac{m_2}{\underline{\alpha}_2}}, \quad (31)$$

where $\underline{\alpha}_k = t_k \tanh^{-1}(\alpha_k/t_k)$, $k = 1, 2$.

Proof. Define $g(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f(\mathbf{x} \oplus \mathbf{n} \otimes \alpha)$. Bearing in mind that

$$g(\mathbf{x} \oplus \alpha) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f((\mathbf{x} \oplus \mathbf{n} \otimes \alpha) \oplus \alpha) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f(\mathbf{x} \oplus (\mathbf{n} + 1) \otimes \alpha) = g(\mathbf{x})$$

it follows that g is an α -periodic function. Moreover, since f is integrable, the function g is defined almost everywhere and integrable over $[0, \alpha_1] \times [0, \alpha_2]$ and satisfies

$$\|g\|_{L^1(Q_\alpha, \mathbb{H})} \leq \|f\|_{L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}.$$

Therefore, we can expand g in terms of its quaternion hyperbolic Fourier series given by

$$g(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^2} e^{2\pi i \underline{x}_1 \frac{m_1}{\underline{\alpha}_1}} a_{\mathbf{m}} e^{2\pi j \underline{x}_2 \frac{m_2}{\underline{\alpha}_2}}, \quad (32)$$

where the coefficients $a_{\mathbf{m}}$ are computed as

$$a_{\mathbf{m}} = \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \int_0^{\alpha_1} \int_0^{\alpha_2} e^{-2\pi i \underline{x}_1 \frac{m_1}{\underline{\alpha}_1}} g(\mathbf{x}) e^{-2\pi j \underline{x}_2 \frac{m_2}{\underline{\alpha}_2}} d\mu(\mathbf{x}). \quad (33)$$

Bearing in mind that (33) is an average integral, the normalization constant that appears outside the integral comes from

$$\int_0^{\alpha_1} \int_0^{\alpha_2} 1 d\mu(\mathbf{x}) = t_1 \tanh^{-1} \left(\frac{x_1}{t_1} \right) \Big|_{x_1=0}^{x_1=\alpha_1} t_2 \tanh^{-1} \left(\frac{x_2}{t_2} \right) \Big|_{x_2=0}^{x_2=\alpha_2} = \underline{\alpha}_1 \underline{\alpha}_2.$$

Therefore

$$\begin{aligned} a_{\mathbf{m}} &= \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \int_0^{\alpha_1} \int_0^{\alpha_2} e^{-2\pi i \underline{x}_1 \frac{m_1}{\underline{\alpha}_1}} \sum_{\mathbf{n} \in \mathbb{Z}^2} f(\mathbf{x} \oplus \mathbf{n} \otimes \boldsymbol{\alpha}) e^{-2\pi j \underline{x}_2 \frac{m_2}{\underline{\alpha}_2}} d\mu(\mathbf{x}) \\ &= \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \sum_{\mathbf{n} \in \mathbb{Z}^2} \int_0^{\alpha_1} \int_0^{\alpha_2} e^{-2\pi i \underline{x}_1 \frac{m_1}{\underline{\alpha}_1}} f(\mathbf{x} \oplus \mathbf{n} \otimes \boldsymbol{\alpha}) e^{-2\pi j \underline{x}_2 \frac{m_2}{\underline{\alpha}_2}} d\mu(\mathbf{x}). \end{aligned}$$

Making the change of variables $\mathbf{x} \oplus \mathbf{n} \otimes \boldsymbol{\alpha} = \mathbf{y}$, we get

$$\begin{aligned} a_{\mathbf{m}} &= \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \sum_{\mathbf{n} \in \mathbb{Z}^2} \int_{n_1 \otimes \alpha_1}^{(n_1+1) \otimes \alpha_1} \int_{n_2 \otimes \alpha_2}^{(n_2+1) \otimes \alpha_2} e^{-2\pi i \underline{y}_1 \frac{m_1}{\underline{\alpha}_1}} e^{-2\pi j \underline{n}_1 \otimes \alpha_1 \frac{m_1}{\underline{\alpha}_1}} f(\mathbf{y}) \\ &\quad \times e^{-2\pi j \underline{y}_2 \frac{m_2}{\underline{\alpha}_2}} e^{-2\pi j \underline{n}_2 \otimes \alpha_2 \frac{m_2}{\underline{\alpha}_2}} d\mu(\mathbf{y}). \end{aligned}$$

Since

$$\begin{aligned} \underline{n_k \otimes \alpha_k} &= t_k \tanh^{-1} \left(\frac{n_k \otimes \alpha_k}{t_k} \right) \\ &= t_k \tanh^{-1} \left(\frac{t_k \tanh(n_k \tanh^{-1}(\alpha_k/t_k))}{t_k} \right) \\ &= n_k \underline{\alpha}_k, \quad k = 1, 2 \end{aligned}$$

we obtain

$$e^{-2\pi q \underline{n_k \otimes \alpha_k} \frac{m_k}{\underline{\alpha}_k}} = e^{-2\pi q n_k m_k} = 1, \quad k = 1, 2.$$

Hence

$$\begin{aligned} a_{\mathbf{m}} &= \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi i \underline{y}_1 \frac{m_1}{\underline{\alpha}_1}} f(\mathbf{y}) e^{-2\pi j \underline{y}_2 \frac{m_2}{\underline{\alpha}_2}} d\mu(\mathbf{y}) \\ &= \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \mathcal{F}_{QH}(f) \left(\frac{m_1}{\underline{\alpha}_1}, \frac{m_2}{\underline{\alpha}_2} \right). \end{aligned}$$

Finally, from (32) we get

$$g(\mathbf{x}) = \frac{1}{\underline{\alpha}_1 \underline{\alpha}_2} \sum_{\mathbf{m} \in \mathbb{Z}^2} e^{2\pi i \underline{x}_1 \frac{m_1}{\underline{\alpha}_1}} \mathcal{F}_{QH}(f) \left(\frac{m_1}{\underline{\alpha}_1}, \frac{m_2}{\underline{\alpha}_2} \right) e^{2\pi j \underline{x}_2 \frac{m_2}{\underline{\alpha}_2}},$$

which gives the desired result. $\square$

It turns out that formula (31) also works for $f \in L^1(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, provided that both infinite sums are absolutely convergent. Furthermore, we can change $\oplus$ to $\ominus$ on the right-hand side of (31) due to the symmetry of the lattice $\mathbb{Z}^2$.

The Wiener space arises naturally in the treatment of periodic functions.

Definition 3.2 (Wiener space). *The Wiener space $W = W(\mathbb{R}_{t_1, t_2}^2)$ is the space of functions $g \in L^\infty(\mathbb{R}_{t_1, t_2}^2)$ such that*

$$\|g\|_W = \sum_{\mathbf{n} \in \mathbb{Z}^2} \operatorname{ess\,sup}_{\mathbf{x} \in Q} |g(\mathbf{x} \ominus \tilde{\mathbf{n}})| < \infty, \quad (34)$$

where $\tilde{\mathbf{n}} = (t_1 \tanh(n_1/t_1), t_2 \tanh(n_2/t_2))$.

Functions in W are locally bounded and are globally in $\ell^1(\mathbb{H})$. This space can be generalized to the so-called amalgam spaces by using norms to control local and global behaviour independently (see [17, 30] and elsewhere).

Using the characteristic function of a set, the norm (34) can be written as

$$\|g\|_W = \sum_{\mathbf{n} \in \mathbb{Z}^2} \|T_{\tilde{\mathbf{n}}} g \chi_Q\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)}. \quad (35)$$

The Wiener space contains all bounded functions with compact support and is therefore a dense subspace of $L^p(\mathbb{R}_{t_1, t_2}^2)$ for each $1 \leq p < \infty$.

Before we proceed further, we give a fundamental result about the boundedness of a function in the Wiener space, which we formulate in a preliminary lemma.

Lemma 3.3. *If $g \in W(\mathbb{R}_{t_1, t_2}^2)$, then*

$$\operatorname{ess\,sup}_{\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2} \sum_{\mathbf{n} \in \mathbb{Z}^2} |g(\mathbf{x} \ominus \mathbf{n} \otimes \boldsymbol{\alpha})| \leq \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \|g\|_W.$$

Proof. First we observe that any interval inside $\mathbb{R}_{t_k}$, $k = 1, 2$, with unit hyperbolic length $\tilde{1} = t_k \tanh(1/t_k)$ contains at most $\underline{\alpha}_k^{-1} + 1$ distinct points of minimum distance α_k between each other. In fact, being l the number of such points, we have the relation

$$\alpha_k = t_k \tanh\left(\frac{1}{(l-1)t_k}\right)$$

from which we get

$$t_k \tanh^{-1}\left(\frac{\alpha_k}{t_k}\right) = \frac{1}{l-1} \Leftrightarrow \underline{\alpha}_k = \frac{1}{l-1} \Leftrightarrow l = \underline{\alpha}_k^{-1} + 1.$$

Thus, the rectangle $Q \oplus \tilde{\mathbf{m}} = [\widetilde{m_1}, \widetilde{m_1} + 1] \times [\widetilde{m_2}, \widetilde{m_2} + 1]$, $\mathbf{m} \in \mathbb{Z}^2$, contains at most $(\underline{\alpha}_1^{-1} + 1)(\underline{\alpha}_2^{-1} + 1)$ points of the form $\mathbf{x} \oplus \mathbf{n} \otimes \boldsymbol{\alpha}$, $n \in \mathbb{Z}^2$, independently of $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$. Therefore,

$$\begin{aligned} \sum_{\mathbf{n} \in \mathbb{Z}^2} |g(\mathbf{x} \ominus \mathbf{n} \otimes \boldsymbol{\alpha})| &\leq \sum_{\mathbf{m} \in \mathbb{Z}^2} \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \sup_{\mathbf{x} \ominus \mathbf{n} \otimes \boldsymbol{\alpha} \in Q \oplus \tilde{\mathbf{m}}} |g(\mathbf{x} \ominus \mathbf{n} \otimes \boldsymbol{\alpha})| \\ &= \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \sum_{\mathbf{m} \in \mathbb{Z}^2} \|T_{\tilde{\mathbf{m}}} g \chi_Q\|_\infty \\ &= \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \|g\|_W. \end{aligned}$$

□

4. The structure of quaternionic hyperbolic Gabor frames

In the previous section, we explored the reconstruction of a signal from its continuous QHWT. For practical applications, however, it is crucial to translate these results into a discrete setting, which involves expanding the signal using a countable subset of space-frequency shifts of the window function. This transition requires replacing integrals with sums and utilizing coefficients on a sufficiently dense lattice. This approach leads us to the concept of *quaternionic hyperbolic Gabor frames*.

In what follows, let $g, \gamma \in L^2(\mathbb{R}_{t_1, t_2}^2)$ be nonzero real-valued window functions and let the lattice parameters $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_{t_1, t_2}^2$ be such that $0 < \alpha_1 < t_1$ and $0 < \alpha_2 < t_2$, and $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ with $\beta_1, \beta_2 > 0$. For convenience, we use the intrinsic variable $\tilde{\beta} = (\widetilde{1/\beta_1}, \widetilde{1/\beta_2}) \in \mathbb{R}_{t_1, t_2}^2$, where $\widetilde{1/\beta_k} = t_k \tanh(1/(\beta_k t_k))$, $k = 1, 2$.

The following statements contain the definitions of a quaternionic hyperbolic Gabor system and correlation function.

Definition 4.1 (Quaternionic hyperbolic Gabor frame). *The set of hyperbolic space-frequency shifts defined by*

$$\mathcal{G}(g, \alpha, \beta) = \{M_{\beta \mathbf{m}} T_{\mathbf{n} \otimes \alpha} g : \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2\},$$

where $\beta \mathbf{m} = (\beta_1 m_1, \beta_2 m_2)$, is called a quaternionic hyperbolic Gabor system. It is a frame for $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ if there exist real constants $A, B > 0$ such that

$$A \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2 \leq \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2} |(f, M_{\beta \mathbf{m}} T_{\mathbf{n} \otimes \alpha} g)|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2 \leq B \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2$$

for all $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$.

Definition 4.2 (Correlation function). *Let $\mathbf{m} \in \mathbb{Z}^2$. The function $G_{\mathbf{m}}$ defined by*

$$G_{\mathbf{m}}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} g(\mathbf{x} \ominus \mathbf{n} \otimes \alpha \ominus \mathbf{m} \otimes \tilde{\beta}) \gamma(\mathbf{x} \ominus \mathbf{n} \otimes \alpha) \quad (36)$$

is called the correlation function between the window functions g and γ .

The following result gives a boundedness property of the correlation function in $L^\infty(\mathbb{R}_{t_1, t_2}^2)$.

Lemma 4.1. *If $g, \gamma \in W(\mathbb{R}_{t_1, t_2}^2)$, then $G_{\mathbf{m}} \in L^\infty(\mathbb{R}_{t_1, t_2}^2)$. Further, we have*

$$\sum_{\mathbf{m} \in \mathbb{Z}^2} \|G_{\mathbf{m}}\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \leq \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) (2\lfloor \beta_1 \rfloor + 1)(2\lfloor \beta_2 \rfloor + 1) \|g\|_W \|\gamma\|_W.$$

Proof. Since $\|T_{\mathbf{m} \otimes \tilde{\beta}} g \gamma\|_W \leq \|g\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \|\gamma\|_W$, it follows that $T_{\mathbf{m} \otimes \tilde{\beta}} g \gamma \in W(\mathbb{R}_{t_1, t_2}^2)$. Applying Lemma 3.3 to $T_{\mathbf{m} \otimes \tilde{\beta}} g \gamma$, we get

$$\|G_{\mathbf{m}}\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \leq \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \|T_{\mathbf{m} \otimes \tilde{\beta}} g \gamma\|_W.$$

Hence,

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbb{Z}^2} \|G_{\mathbf{m}}\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} &\leq \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \sum_{\mathbf{m} \in \mathbb{Z}^2} \|T_{\mathbf{m} \otimes \tilde{\beta}} g \gamma\|_W \\ &= \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \sum_{\mathbf{m} \in \mathbb{Z}^2} \sum_{\mathbf{n} \in \mathbb{Z}^2} \|(T_{\tilde{\mathbf{n}}} T_{\mathbf{m} \otimes \tilde{\beta}} g \chi_Q)(T_{\tilde{\mathbf{n}}} \gamma \chi_Q)\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \\ &\leq \left(\frac{1}{\underline{\alpha}_1} + 1\right) \left(\frac{1}{\underline{\alpha}_2} + 1\right) \sum_{\mathbf{n} \in \mathbb{Z}^2} \left(\sum_{\mathbf{m} \in \mathbb{Z}^2} \|T_{\tilde{\mathbf{n}}} T_{\mathbf{m} \otimes \tilde{\beta}} g \chi_Q\|_\infty \right) \|T_{\tilde{\mathbf{n}}} \gamma \chi_Q\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)}. \quad (37) \end{aligned}$$

We proceed to estimate each term in the inner sum as follows:

$$\|T_{\tilde{\mathbf{n}}} T_{\mathbf{m} \otimes \tilde{\beta}} g \chi_Q\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} = \operatorname{ess\,sup}_{\mathbf{x} \in \mathbf{m} \otimes \tilde{\beta} \oplus \tilde{\mathbf{n}} \oplus Q} |g(\mathbf{x})| \leq \sum_{\ell \in I_{\mathbf{m}}} \|T_{\tilde{\ell}} g \chi_Q\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)},$$

where

$$I_{\mathbf{m}} = \{\ell \in \mathbb{Z}^2 : \mathbf{m} \otimes \tilde{\beta} \oplus \tilde{\mathbf{n}} \oplus Q \cap \tilde{\ell} \oplus Q \neq \emptyset\}.$$

To count the cardinality of the set $I_{\mathbf{m}}$, we first solve the condition $0 \leq |m_k \otimes \widetilde{1/\beta_k}| \leq \widetilde{1_k}$ for $m_k \in \mathbb{Z}$, where $\widetilde{1_k} = t_k \tanh(1/t_k)$, $k = 1, 2$. After some manipulations, we find

$$0 \leq |m_k \otimes \widetilde{1/\beta_k}| \leq \widetilde{1_k} \Leftrightarrow 0 \leq |m_k| \tanh^{-1}\left(\frac{\widetilde{1/\beta_k}}{t_k}\right) \leq \frac{1}{t_k} \Leftrightarrow 0 \leq \frac{|m_k|}{\beta_k} \leq 1,$$

which is equivalent to

$$0 \leq |m_k| \leq \beta_k.$$

Since the last conditions are now stated in $\mathbb{R}$, there are $2\lfloor \beta_k \rfloor + 1$ indices that satisfy these conditions. Consequently, the cardinality of the set $I_{\mathbf{m}}$ is $(2\lfloor \beta_1 \rfloor + 1)(2\lfloor \beta_2 \rfloor + 1)$. Therefore, we get

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbb{Z}^2} \|T_{\tilde{\mathbf{n}}} T_{\mathbf{m} \otimes \tilde{\beta}} g \chi_Q\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} &\leq (2\lfloor \beta_1 \rfloor + 1)(2\lfloor \beta_2 \rfloor + 1) \sum_{\ell \in \mathbb{Z}^2} \|T_{\tilde{\ell}} g \chi_Q\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \\ &= (2\lfloor \beta_1 \rfloor + 1)(2\lfloor \beta_2 \rfloor + 1) \|g\|_W \end{aligned} \quad (38)$$

independently of $\mathbf{n} \in \mathbb{Z}^2$. The outer sum in (37) gives $\|\gamma\|_W$. Finally, putting the estimate (38) into (37), the statement of the result follows. $\square$

4.1. Boundedness results for the hyperbolic quaternionic frame operator

Building upon the reconstruction formula (29) and replacing the integrals with sums, we next define the quaternionic hyperbolic frame operator generated by a general lattice.

Definition 4.3 (Quaternionic hyperbolic frame operator). *Let $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ and let $g, \gamma \in L^2(\mathbb{R}_{t_1, t_2}^2)$ be non-zero window functions. The quaternionic hyperbolic frame operator $S_{g, \gamma}^H$ associated with the analysis window g and the synthesis window γ is defined by*

$$S_{g, \gamma}^H f = \sum_{\mathbf{m} \in \mathbb{Z}^2} \sum_{\mathbf{n} \in \mathbb{Z}^2} M'_{-\mathbf{x}; \beta \mathbf{m}}(E(M_{\beta \mathbf{m}}(f T_{\mathbf{n} \otimes \alpha} g))) T_{\mathbf{n} \otimes \alpha} \gamma, \quad (39)$$

where M' is defined in (19).

The following theorem gives a representation of the quaternionic hyperbolic frame operator purely in terms of the correlation function $G_{\mathbf{m}}(\mathbf{x})$ defined by (36).

Theorem 4.1 (Walnut's representation). *Let $g, \gamma \in W$. For any $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ the operator $S_{g, \gamma}^H$ can be written as*

$$S_{g, \gamma}^H f = (\beta_1 \beta_2)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}} T_{\mathbf{m} \otimes \tilde{\beta}} f.$$

Further, this series converges absolutely in $L^2(\mathbb{R}_{t_1, t_2}^2)$.

Proof. For every $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, we have

$$E(M_{\beta \mathbf{m}}(f T_{\mathbf{n} \otimes \alpha} g)) = \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi i x_1 \beta_1 m_1} f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{n} \otimes \alpha) e^{-2\pi i x_2 \beta_2 m_2} d\mu(\mathbf{x}) \in \ell^2(\mathbb{H}).$$

By the quaternionic hyperbolic Poisson summation formula (31) for the ts-QHWFT, it follows that

$$\begin{aligned} \xi_n &= \sum_{\mathbf{m} \in \mathbb{Z}^2} M'_{-\mathbf{x}; \beta \mathbf{m}} E(M_{\beta \mathbf{m}}(f T_{\mathbf{n} \otimes \alpha} g)) \\ &= (\beta_1 \beta_2)^{-1} \sum_{n \in \mathbb{Z}^2} (f T_{\mathbf{n} \otimes \alpha} g)(\mathbf{x} \ominus \mathbf{m} \otimes \tilde{\beta}). \end{aligned}$$

Therefore, the quaternionic frame operator becomes

$$S_{g, \gamma}^H f = \sum_{\mathbf{n} \in \mathbb{Z}^2} \left(\sum_{\mathbf{m} \in \mathbb{Z}^2} (\beta_1 \beta_2)^{-1} f(\mathbf{x} \ominus \mathbf{m} \otimes \tilde{\beta}) g(\mathbf{x} \ominus \mathbf{n} \otimes \alpha \ominus \mathbf{m} \otimes \tilde{\beta}) \right) \gamma(\mathbf{x} \ominus \mathbf{n} \otimes \alpha).$$

Assuming that f has compact support, the sum over $\mathbf{m}$ is finite for a fixed $x \in \mathbb{R}_{t_1, t_2}^2$, so we can interchange the order of summation. Thus, we get

$$\begin{aligned} S_{g, \gamma}^H f &= (\beta_1 \beta_2)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^2} \left(\sum_{\mathbf{n} \in \mathbb{Z}^2} g(\mathbf{x} \ominus \mathbf{n} \otimes \alpha \ominus \mathbf{m} \otimes \tilde{\beta}) \gamma(\mathbf{x} \ominus \mathbf{n} \otimes \alpha) \right) f(\mathbf{x} \ominus \mathbf{m} \otimes \tilde{\beta}) \\ &= (\beta_1 \beta_2)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}} T_{\mathbf{m} \otimes \tilde{\beta}} f, \end{aligned}$$

which shows that the statement holds for functions with compact support.

The extension to L^2 follows by density arguments. This completes the proof of the theorem. $\square$

From Theorem 4.1 and Lemma 4.1, we can obtain the L^p -boundedness of the quaternionic frame operator for $1 \leq p \leq \infty$. We have

$$\begin{aligned} \|S_{g, \gamma}^H f\|_{L^p(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} &\leq (\beta_1 \beta_2)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^2} \|G_{\mathbf{m}} T_{\mathbf{m} \otimes \tilde{\beta}} f\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} \\ &\leq (\beta_1 \beta_2)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^2} \|G_{\mathbf{m}}\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \|T_{\mathbf{m} \otimes \tilde{\beta}} f\|_{L^p(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} \\ &\leq (\beta_1 \beta_2)^{-1} \left(\frac{1}{\underline{\alpha}_1} + 1 \right) \left(\frac{1}{\underline{\alpha}_2} + 1 \right) (2\lfloor \beta_1 \rfloor + 1)(2\lfloor \beta_2 \rfloor + 1) \|g\|_W \|\gamma\|_W \|f\|_{L^p(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \end{aligned}$$

Our present concern is with the orthogonal properties of the quaternionic hyperbolic frame operator in the space $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$.

Theorem 4.2. Let $g, \gamma \in W$. Let $Q_{\tilde{\beta}}$ be a two-dimensional rectangle with side-lengths $1/\widetilde{\beta_1}$ and $1/\widetilde{\beta_2}$ and let $G_{\ell, \mathbf{n}}$ be given by

$$G_{\ell, \mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{r} \in \mathbb{Z}^2} g(\mathbf{x} \ominus \ell \otimes \tilde{\beta} \ominus \mathbf{r} \otimes \alpha) \gamma(\mathbf{x} \ominus \mathbf{n} \otimes \tilde{\beta} \ominus \mathbf{r} \otimes \alpha).$$

For all $f, h \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$, we have

$$(S_{g, \gamma}^H f, h)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = (\beta_1 \beta_2)^{-1} \sum_{\ell, \mathbf{n} \in \mathbb{Z}^2} \int_{Q_{\tilde{\beta}}} G_{\ell, \mathbf{n}}(\mathbf{x}) T_{\ell \otimes \tilde{\beta}} f(\mathbf{x}) \overline{T_{\mathbf{n} \otimes \tilde{\beta}} h(\mathbf{x})} d\mu(\mathbf{x}). \quad (40)$$

Proof. First, we assume that f and h are bounded and have compact support. Hence, the functions $T_{\mathbf{m} \otimes \tilde{\beta}} f$ and $T_{\mathbf{m} \otimes \tilde{\beta}} h$ with $\mathbf{m} \in \mathbb{Z}^2$ have compact support. By Theorem 4.1 and Lemma 3.1, we get

$$\begin{aligned} & (S_{g,\gamma}^H f, h)_{L^2(\mathbb{R}_{t_1,t_2}^2, \mathbb{H})} \\ &= (\beta_1 \beta_2)^{-1} \int_{\mathbb{R}_{t_1,t_2}^2} \sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}}(\mathbf{x}) f(\mathbf{x} \ominus \mathbf{m} \oplus \tilde{\beta}) \overline{h(\mathbf{x})} d\mu(\mathbf{x}) \\ &= (\beta_1 \beta_2)^{-1} \int_{Q_{\tilde{\beta}}} \sum_{\mathbf{n} \in \mathbb{Z}^2} \sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}}(\mathbf{x} \ominus \mathbf{n} \otimes \tilde{\beta}) f(\mathbf{x} \ominus (\mathbf{m} + \mathbf{n}) \otimes \tilde{\beta}) \overline{h(\mathbf{x} \ominus \mathbf{n} \otimes \tilde{\beta})} d\mu(\mathbf{x}). \end{aligned}$$

Due to the compact support of the functions, the sums over $\mathbf{m}$ and $\mathbf{n}$ are finite. Making the change of variable $\ell = \mathbf{m} + \mathbf{n}$ and using the fact that $G_{\mathbf{m}}(\mathbf{x} \ominus \mathbf{n} \otimes \tilde{\beta}) = G_{\mathbf{m}+\mathbf{n},\mathbf{n}}(\mathbf{x})$, we obtain

$$(S_{g,\gamma}^H f, h)_{L^2(\mathbb{R}_{t_1,t_2}^2, \mathbb{H})} = (\beta_1 \beta_2)^{-1} \int_{Q_{\tilde{\beta}}} \sum_{\mathbf{n}, \ell \in \mathbb{Z}^2} G_{\ell,\mathbf{n}}(\mathbf{x}) T_{\ell \otimes \tilde{\beta}} f(\mathbf{x}) \overline{T_{\mathbf{n} \otimes \tilde{\beta}} h(\mathbf{x})} d\mu(\mathbf{x}). \quad (41)$$

For the extension of (41) to arbitrary $f, g \in L^2(\mathbb{R}_{t_1,t_2}^2, \mathbb{H})$, we introduce the matrix operator $G(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}_{t_1,t_2}^2$, acting on finite sequences $\mathbf{c} = (c_\ell)_{\ell \in \mathbb{Z}^2}$ by the matrix multiplication

$$(G(\mathbf{x})\mathbf{c})_{\mathbf{n}} = \sum_{\ell} G_{\ell,\mathbf{n}}(\mathbf{x}) c_\ell.$$

If $g, \gamma \in W$, then each $G(\mathbf{x})$ satisfies the conditions of Schur's Test [21, Lemma 6.2.1]. Thus, we get

$$\begin{aligned} \sum_{\ell} |G_{\ell,\mathbf{n}}(\mathbf{x})| &\leq \sum_{\ell} \sum_{\mathbf{r}} |g(\mathbf{x} \ominus \ell \otimes \tilde{\beta} \ominus \mathbf{r} \otimes \alpha)| |\gamma(\mathbf{x} \ominus \mathbf{n} \otimes \tilde{\beta} \ominus \mathbf{r} \otimes \alpha)| \\ &= \sum_{\mathbf{r}} \left(\sum_{\ell} |g(\mathbf{x} \ominus \ell \otimes \tilde{\beta} \ominus \mathbf{r} \otimes \alpha)| \right) |\gamma(\mathbf{x} \ominus \mathbf{n} \otimes \tilde{\beta} \ominus \mathbf{r} \otimes \alpha)| \\ &\leq \left(\frac{1}{\underline{\alpha}_1} + 1 \right) \left(\frac{1}{\underline{\alpha}_2} + 1 \right) (\beta_1 + 1)(\beta_2 + 1) \|g\|_W \|\gamma\|_W. \end{aligned}$$

A similar estimate holds for $\sum_{\mathbf{n}} |G_{\ell,\mathbf{n}}(\mathbf{x})|$. Hence, $G(\mathbf{x})$ defines a bounded operator on ℓ^p , $1 \leq p \leq \infty$. If $f, h \in L^2(\mathbb{R}_{t_1,t_2}^2)$, then both sequences $\{f(\mathbf{x} \ominus \ell \otimes \tilde{\beta}), \ell \in \mathbb{Z}^2\}$ and $\{h(\mathbf{x} \ominus \ell \otimes \tilde{\beta}), \ell \in \mathbb{Z}^2\}$ are in $\ell^2(\mathbb{H})$ for almost all $\mathbf{x} \in \mathbb{R}_{t_1,t_2}^2$. Thus, the matrix representation holds. $\square$

Remark 4.1. If $g, \gamma \in L^2(\mathbb{R}_{t_1,t_2}^2)$, then $T_{\ell \otimes \tilde{\beta}} g, T_{\mathbf{n} \otimes \tilde{\beta}} \gamma \in L^1(\mathbb{R}_{t_1,t_2}^2)$ and so $G_{\ell,\mathbf{n}}(\mathbf{x}) \in L^1(Q_\alpha)$ for all $\ell, \mathbf{n} \in \mathbb{Z}^2$. Therefore, (40) holds whenever f and h are bounded and have compact support inside $\mathbb{R}_{t_1,t_2}^2$. Further, if $g = \gamma$, then $G(\mathbf{x})$ defines a positive operator, i.e., $(G(\mathbf{x})\mathbf{c}, \mathbf{c})_{\ell^2} \geq 0$ for all $\mathbf{c} = (c_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in \ell^2(\mathbb{H})$.

In the following, we prove some results about the boundedness and invertibility of the hyperbolic quaternionic Gabor frames.

Theorem 4.3. Let $g \in L^2(\mathbb{R}_{t_1,t_2}^2)$ and let I denote the identity operator in $\ell^2(\mathbb{H})$. Then

- (a) $S_{g,g}^H$ is invertible on $L^2(\mathbb{R}_{t_1,t_2}^2, \mathbb{H})$ if and only if there exists a constant $a > 0$ such that $G(\mathbf{x}) \geq aI$ for almost all $\mathbf{x} \in \mathbb{R}_{t_1,t_2}^2$.
- (b) $S_{g,g}^H$ is a bounded operator on $L^2(\mathbb{R}_{t_1,t_2}^2, \mathbb{H})$ if and only if there exists a constant $b > 0$ such that $G(\mathbf{x}) \leq bI$ for almost all $\mathbf{x} \in \mathbb{R}_{t_1,t_2}^2$.

Proof. For simplicity, we prove only the first statement. The proof of the second statement is similar.

Assume that $G(\mathbf{x}) \geq aI$. Then for every $f \in L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ with compact support, we have

$$\left| \sum_{\ell, \mathbf{n}} G_{\ell, \mathbf{n}} (T_\ell f) \overline{T_{\mathbf{n}} f} \right| \geq a \sum_{\mathbf{n}} |T_{\mathbf{n}} f|^2.$$

Integrating over $Q_{\tilde{\beta}}$, by Theorem 4.2, we obtain

$$\begin{aligned} (S_{g, g}^H f, f)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} &\geq a(\beta_1 \beta_2)^{-1} \int_{Q_{\tilde{\beta}}} \sum_{\mathbf{n}} |T_{\mathbf{n}} f(\mathbf{x})|^2 d\mu(\mathbf{x}) \\ &= a(\beta_1 \beta_2)^{-1} \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2. \end{aligned}$$

By using density arguments, the statement holds for all $f \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. Hence, $S_{g, g}^H$ is invertible.

Conversely, if $S_{g, g}^H$ is not invertible, then there exists a sequence of bounded functions $(f_k)_{k \in \mathbb{N}}$ with compact support such that

$$(S_{g, g}^H f_k, f_k)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} < (1/k) \|f_k\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2.$$

By Theorem 4.2, it follows that

$$\int_{Q_{\tilde{\beta}}} \left(\frac{1}{k} \sum_{\mathbf{m}} |f_k(\mathbf{x} \ominus \mathbf{m})|^2 - (\beta_1 \beta_2)^{-1} \sum_{\mathbf{m}, \mathbf{n}} G_{\mathbf{m}, \mathbf{n}} f_k(\mathbf{x} \ominus \mathbf{m}) \overline{f_k(\mathbf{x} \ominus \mathbf{n})} \right) d\mu(\mathbf{x}) > 0.$$

Therefore, there exists a sequence of sets $E_k \subseteq Q_{\tilde{\beta}}$ with positive measure and such that for $\mathbf{x} \in E_k$ holds

$$\frac{1}{k} \sum_{\mathbf{m}} |f_k(\mathbf{x} \ominus \mathbf{m})|^2 - \sum_{\mathbf{m}, \mathbf{n}} G_{\mathbf{m}, \mathbf{n}} f_k(\mathbf{x} \ominus \mathbf{m}) \overline{f_k(\mathbf{x} \ominus \mathbf{n})} > 0.$$

Hence, we get

$$\inf_{\|c\|_{\ell^2}^2 \leq 1} (G(\mathbf{x}) \mathbf{c}, \mathbf{c})_{\ell^2} < \frac{\beta_1 \beta_2}{k}$$

for $\mathbf{x} \in E_k$. But a uniform inequality $aI \leq G(\mathbf{x})$ cannot hold for almost all $\mathbf{x}$. The result follows. $\square$

Lemma 4.2. *If $S_{g, \gamma}^H$ is bounded, then*

$$(S_{g, \gamma}^H f, h)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} = (\beta_1 \beta_2)^{-1} \left(\sum_{\mathbf{m}} G_{\mathbf{m}} T_{\mathbf{m} \otimes \tilde{\beta}}, h \right)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}$$

for all $\mathbf{n} \in \mathbb{Z}^2$ and all $f, h \in L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ with compact support.

Lemma 4.3. *If $S_{g, \gamma}^H$ is bounded, then*

$$\|G_{\mathbf{m}}\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \leq \beta_1 \beta_2 \|S_{g, \gamma}^H\|_{op}$$

for all $\mathbf{m} \in \mathbb{Z}^2$, where $\|\cdot\|_{op}$ denotes the operator norm.

Proof. Let $f, h \in L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ with support in $Q_{\tilde{\beta}}$ and $\mathbf{n}, \ell \in \mathbb{Z}^2$. Due to the boundedness of $S_{g, \gamma}^H$, we have

$$|(S_{g, \gamma}^H T_{\ell \otimes \tilde{\beta}} f, T_{\mathbf{n} \otimes \tilde{\beta}} h)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}| \leq \|S_{g, \gamma}^H\|_{\text{op}} \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} \|h\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \quad (42)$$

By Lemma 4.2, we obtain

$$\begin{aligned} (S_{g, \gamma}^H T_{\ell \otimes \tilde{\beta}} f, T_{\mathbf{n} \otimes \tilde{\beta}} h)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} &= (\beta_1 \beta_2)^{-1} \left(\sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}} T_{\mathbf{m} \otimes \tilde{\beta}} (T_{\ell \otimes \tilde{\beta}} f), T_{\mathbf{n} \otimes \tilde{\beta}} h \right)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} \\ &= (\beta_1 \beta_2)^{-1} \left(\sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}} T_{(\mathbf{m} + \ell) \otimes \tilde{\beta}} f, T_{\mathbf{n} \otimes \tilde{\beta}} h \right)_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}. \end{aligned} \quad (43)$$

Since the supports of $T_{(\mathbf{m} + \ell) \otimes \tilde{\beta}} f$ and $T_{\mathbf{n} \otimes \tilde{\beta}} h$ are generally pairwise disjoint, only the term $\mathbf{m} + \ell = \mathbf{n}$ survives. Combining (42) and (43), we get

$$(\beta_1 \beta_2)^{-1} \left| \int_{Q_{\tilde{\beta}}} G_{\mathbf{n} - \ell}(\mathbf{x} \oplus \mathbf{n} \otimes \tilde{\beta}) f(\mathbf{x}) \overline{h(\mathbf{x})} d\mu(\mathbf{x}) \right| \leq \|S_{g, \gamma}^H\|_{\text{op}} \|f\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})} \|h\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}, \quad (44)$$

for all $f, h \in L^\infty(\mathbb{R}_{t_1, t_2}^2, \mathbb{H}) \subset L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$ and all $\mathbf{n}, \ell \in \mathbb{Z}^2$. A density argument applied to (44) extends the inequality to $f, h \in L^2(Q_{\tilde{\beta}}, \mathbb{H})$, and we have

$$(\beta_1 \beta_2)^{-1} \text{ess sup}_{\mathbf{x} \in Q_{\tilde{\beta}}} \left| G_{\mathbf{m}}(\mathbf{x} \oplus (\mathbf{m} \oplus \ell) \otimes \tilde{\beta}) \right| \leq \|S_{g, \gamma}^H\|_{\text{op}}$$

for all $\ell \in \mathbb{Z}^2$. Thus,

$$\|G_{\mathbf{m}}\|_{L^\infty(\mathbb{R}_{t_1, t_2}^2)} \leq \beta_1 \beta_2 \|S_{g, \gamma}^H\|_{\text{op}}.$$

This completes the proof of the statement. $\square$

4.2. Janssen's representation of the quaternionic hyperbolic frame operator

Walnut's representation of the quaternionic hyperbolic Gabor frame operator treats modulations and translations differently, leading to formulas that lack invariance under the FT. In contrast, Janssen's representation gives a more symmetric representation of the quaternionic hyperbolic Gabor frame operator.

We proceed to compute the ℓ -th discrete QHFT coefficients of $G_{\mathbf{m}}$ using (33):

$$\begin{aligned} \widehat{G_{\mathbf{m}}}(\ell) &= (\underline{\alpha}_1 \ \underline{\alpha}_2)^{-1} \int_{Q_{\alpha}} e^{-2\pi i \underline{x}_1 \frac{l_1}{\alpha_1}} G_{\mathbf{m}}(\mathbf{x}) e^{-2\pi j \underline{x}_2 \frac{l_2}{\alpha_2}} d\mu(\mathbf{x}) \\ &= (\underline{\alpha}_1 \ \underline{\alpha}_2)^{-1} \int_{Q_{\alpha}} \sum_{\mathbf{n} \in \mathbb{Z}^2} e^{-2\pi i \underline{x}_1 \frac{l_1}{\alpha_1}} ((T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma)(\mathbf{x} \ominus \mathbf{n} \otimes \alpha) e^{-2\pi j \underline{x}_2 \frac{l_2}{\alpha_2}} d\mu(\mathbf{x}) \\ &= (\underline{\alpha}_1 \ \underline{\alpha}_2)^{-1} \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi i \underline{x}_1 \frac{l_1}{\alpha_1}} ((T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma)(\mathbf{x}) e^{-2\pi j \underline{x}_2 \frac{l_2}{\alpha_2}} d\mu(\mathbf{x}) \\ &= (\underline{\alpha}_1 \ \underline{\alpha}_2)^{-1} E(M_{\ell/\alpha}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma), \end{aligned}$$

where $(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma$ denotes the pointwise multiplication operator of both functions, i.e.,

$$\mathbf{x} \mapsto ((T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma)(\mathbf{x}) = T_{\mathbf{m} \otimes \tilde{\beta}} g(\mathbf{x}) \gamma(\mathbf{x}). \quad (45)$$

By Lemma 4.2, $G_{\mathbf{m}}(\mathbf{x})$ has the following Fourier expansion in $L^2(Q_{\alpha})$:

$$G_{\mathbf{m}}(\mathbf{x}) = (\underline{\alpha}_1 \ \underline{\alpha}_2)^{-1} \sum_{\ell \in \mathbb{Z}^2} e^{2\pi i \underline{x}_1 \frac{l_1}{\alpha_1}} E(M_{\ell/\alpha}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma) e^{2\pi j \underline{x}_2 \frac{l_2}{\alpha_2}}, \quad (46)$$

which coincides with the quaternionic hyperbolic Poisson summation formula (31).

Since the series in (46) may not converge for general $g, \gamma \in L^2(\mathbb{R}_{t_1, t_2}^2)$, we are going to use condition (A') of Tolimieri and Orr [43].

Definition 4.4 (Condition (A')). *Let the lattice parameters $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_{t_1, t_2}^2$ be such that $0 < \alpha_1 < t_1$ and $0 < \alpha_2 < t_2$, and $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ with $\beta_1, \beta_2 > 0$. A pair (g, γ) of real-valued functions in $L^2(\mathbb{R}_{t_1, t_2}^2)$ satisfies condition (A') for the parameters α, β if*

$$\sum_{\ell, \mathbf{m}} |E(M_{\ell/\alpha}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma)| < \infty.$$

Replacing $G_{\mathbf{m}}(\mathbf{x})$ in Walnut's Representation Theorem 4.1 and using condition (A'), we obtain the following representation of $S_{g, \gamma}^H$.

Theorem 4.4 (Janssen's representation). *Suppose that the pair (g, γ) satisfies condition (A') for a given choice of the parameters α, β . Then*

$$S_{g, \gamma}^H f = (\underline{\alpha_1} \ \underline{\alpha_2} \ \beta_1 \ \beta_2)^{-1} \sum_{\ell, \mathbf{m} \in \mathbb{Z}^2} e^{2\pi i \underline{x_1} \frac{l_1}{\alpha_1}} E(M_{\ell/\alpha}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma) e^{2\pi i \underline{x_2} \frac{l_2}{\alpha_2}} T_{\mathbf{m} \otimes \tilde{\beta}} f, \quad (47)$$

with absolute convergence in operator norm.

While by definition, the quaternionic hyperbolic frame operator (39) is an expansion where f occurs in the frame coefficients, in Janssen's Representation, the quaternionic hyperbolic frame operator is a linear combination of hyperbolic space-frequency shifts of f .

4.3. Wexler-Raz biorthogonality

The following result gives the relativistic counterpart of the Wexler-Raz biorthogonality condition for the quaternionic hyperbolic frame operator.

Theorem 4.5 (Wexler-Raz biorthogonality). *Let $S_{g, \gamma}$ and $S_{\gamma, g}$ be bounded operators on $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. Then the following conditions are equivalent:*

1. $S_{g, \gamma} = S_{\gamma, g} = I$ on $L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$,
2. $(\underline{\alpha_1} \ \underline{\alpha_2} \ \beta_1 \ \beta_2)^{-1} E(M_{\ell/\alpha}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma) = \delta_{\ell, 0} \delta_{\mathbf{m}, 0}$, for all $\ell, \mathbf{m} \in \mathbb{Z}^2$.

Proof. 1. $\Rightarrow$ 2. If $S_{g, \gamma} = S_{\gamma, g} = I$, then for arbitrary $f, h \in L^\infty(Q_{\tilde{\beta}}, \mathbb{H})$ and $\ell, \mathbf{n} \in \mathbb{Z}^2$ we have

$$\begin{aligned} \delta_{\ell, \mathbf{n}}(f, h)_{L^2(Q_{\tilde{\beta}}, \mathbb{H})} &= (S_{g, \gamma} T_{\ell \otimes \tilde{\beta}} f, T_{\mathbf{n} \otimes \tilde{\beta}} h)_{L^2(Q_{\tilde{\beta}}, \mathbb{H})} \\ &= (\beta_1 \beta_2)^{-1} \left(\sum_{\mathbf{m} \in \mathbb{Z}^2} G_{\mathbf{m}} T_{(\mathbf{m} + \ell) \otimes \tilde{\beta}} f, T_{\mathbf{n} \otimes \tilde{\beta}} h \right)_{L^2(Q_{\tilde{\beta}}, \mathbb{H})} \\ &= (\beta_1 \beta_2)^{-1} (G_{\mathbf{n} - \ell} T_{\mathbf{n} \otimes \tilde{\beta}} f, T_{\mathbf{n} \otimes \tilde{\beta}} h)_{L^2(Q_{\tilde{\beta}}, \mathbb{H})} \\ &= (\beta_1 \beta_2)^{-1} ((T_{-\mathbf{n} \otimes \tilde{\beta}} G_{\mathbf{n} - \ell}) f, h)_{L^2(Q_{\tilde{\beta}}, \mathbb{H})}, \end{aligned}$$

where in the third equality we have used the argument that the supports of $T_{(\mathbf{m} + \ell) \otimes \tilde{\beta}} f$ and $T_{\mathbf{n} \otimes \tilde{\beta}} h$ are generally pairwise disjoint, and therefore, only the term $\mathbf{m} + \ell = \mathbf{n}$ survives. Considering density arguments, this equality extends to all $f, h \in L^2(Q_{\tilde{\beta}})$. Therefore, it holds

$$(\beta_1 \beta_2)^{-1} G_{\mathbf{n} - \ell}(\mathbf{x} \oplus \mathbf{n} \otimes \tilde{\beta}) = \delta_{\ell, \mathbf{n}}$$

for almost all $\mathbf{x} \in Q_{\tilde{\beta}}$. In particular, it follows that $(\beta_1\beta_2)^{-1}G_0(\mathbf{x}) = 1$ and $G_{\mathbf{m}}(\mathbf{x}) = 0$ if $\mathbf{m} \neq \mathbf{0}$ for a.e. $\mathbf{x} \in \mathbb{R}_{t_1, t_2}^2$. Since $G_{\mathbf{m}} \in L^\infty(Q_\alpha)$ has the Fourier series (45)

$$G_{\mathbf{m}}(\mathbf{x}) = (\underline{\alpha}_1 \ \underline{\alpha}_2)^{-1} \sum_{\ell \in \mathbb{Z}^2} e^{2\pi i \frac{x_1}{\underline{\alpha}_1} \frac{l_1}{\underline{\alpha}_1}} E(M_{\ell/\underline{\alpha}}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma) e^{2\pi i \frac{x_2}{\underline{\alpha}_2} \frac{l_2}{\underline{\alpha}_2}},$$

by the uniqueness of the Fourier coefficients, we obtain

$$(\underline{\alpha}_1 \ \underline{\alpha}_2 \ \beta_1 \ \beta_2)^{-1} E(M_{\ell/\underline{\alpha}}(T_{\mathbf{m} \otimes \tilde{\beta}} g) \gamma) = \delta_{\ell, 0} \delta_{\mathbf{m}, 0}$$

for all $\ell, \mathbf{m} \in \mathbb{Z}^2$.

Now, we prove the converse $2. \Rightarrow 1$. If 2. holds then the pair of window functions (g, γ) fulfils the condition (A'). Thus, the Janssen's Representation (47) of $S_{g, \gamma}^H$ converges in the operator norm, with $S_{g, \gamma} = I$. This completes the proof of the theorem. $\square$

We end this section with the following theorem.

Theorem 4.6. *Let $g \in L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})$. A quaternionic hyperbolic Gabor system*

$$\mathcal{G}(g, \alpha, \beta) = \{M'_{\mathbf{m}\underline{\alpha}}(\cdot) T_{\mathbf{n} \otimes \tilde{\beta}} g : \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2\}$$

of operators acting on $\ell^2(\mathbb{H})$ is a tight frame if and only if $\mathcal{G}(g, 1/\beta, \tilde{\alpha})$ is an orthonormal system, where $1/\beta = (1/\beta_1, 1/\beta_2)$ and $\tilde{\alpha} = (1/\underline{\alpha}_1, 1/\underline{\alpha}_2)$. Further, the frame bound A satisfies $A = (\underline{\alpha}_1 \ \underline{\alpha}_2 \ \beta_1 \ \beta_2)^{-1} \|g\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2$.

Proof. By hypothesis, we have $(1/A) S_{g, g}^H = I$. Putting $\gamma = (1/A)g$ in 2. of Theorem 4.5, we get

$$(\underline{\alpha}_1 \ \underline{\alpha}_2 \ \beta_1 \ \beta_2)^{-1} \frac{1}{A} E(M_{\ell/\underline{\alpha}}(T_{\mathbf{m} \otimes \tilde{\beta}} g) g) = \delta_{\ell, 0} \delta_{\mathbf{m}, 0}.$$

In particular, for $\ell = \mathbf{n} = \mathbf{0}$, it follows that

$$(\underline{\alpha}_1 \ \underline{\alpha}_2 \ \beta_1 \ \beta_2)^{-1} \frac{1}{A} \int_{\mathbb{R}_{t_1, t_2}^2} |g(\mathbf{x})|^2 d\mu(\mathbf{x}) = 1.$$

Thus, $A = (\underline{\alpha}_1 \ \underline{\alpha}_2 \ \beta_1 \ \beta_2)^{-1} \|g\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2$ for the frame bound.

Conversely, if $\mathcal{G}(g, 1/\beta, \tilde{\alpha})$ is an orthonormal system, by Janssen's Representation, we obtain

$$S_{g, g}^H = (\underline{\alpha}_1 \ \underline{\alpha}_2 \ \beta_1 \ \beta_2)^{-1} \|g\|_{L^2(\mathbb{R}_{t_1, t_2}^2, \mathbb{H})}^2 I.$$

Thus, $\mathcal{G}(g, \alpha, \beta)$ is a tight frame. The statement of the theorem is thus established. $\square$

5. Application of the ts-QWFT

This section highlights an application of the ts-QHWFT which underscores the relevance of the proposed transform, particularly in the analysis of two-dimensional linear space-varying hyperbolic systems (SVHSs). We can consider the ts-QHWFT as an SVHS band-pass filter element of a filter-bank spectrum analyzer. As a result, the spectrum obtained by the ts-QHWFT can be interpreted as the output of such an SVHS band-pass filter element. An example is given to illustrate the applicability of our results.

Definition 5.1. Let f be a quaternion-valued input signal in $\mathbb{R}_{t_1, t_2}^2$. Consider a two-dimensional SVHS denoting the quaternion impulse response filter. The output response r of the system is defined by

$$r(\mathbf{b}) = \int_{\mathbb{R}_{t_1, t_2}^2} f(\mathbf{x}) h(\mathbf{b}, \mathbf{b} \ominus \mathbf{x}) d\mu(\mathbf{x}),$$

where $\mathbf{b} \in \mathbb{R}_{t_1, t_2}^2$. Further, the transfer function H of the SVHS impulse response filter h is defined as

$$H(\omega, \mathbf{b}) = \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi \mathbf{i} \underline{x}_1 \omega_1} h(\mathbf{b}, \mathbf{x}) e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2} d\mu(\mathbf{x}).$$

The following theorem relates the ts-QHWFT to the output of a two-dimensional SVHS band-pass filter.

Theorem 5.1. Let g be a non-zero real-valued window function and consider a two-dimensional SVHS with quaternion impulse response filter h_1 with respect to g defined as

$$h_1(\mathbf{b}, \mathbf{x}) = e^{-2\pi \mathbf{i} \underline{b}_1 (\underline{b}_1 \ominus \underline{x}_1)} g(\ominus \mathbf{x}) e^{-2\pi \mathbf{j} \underline{b}_2 (\underline{b}_2 \ominus \underline{x}_2)}, \quad \mathbf{b}, \mathbf{x} \in \mathbb{R}_{t_1, t_2}^2. \quad (48)$$

Let the input to this system be the hyperbolic quaternion signal f such that $f\mathbf{i} = \mathbf{i}f$. Then the output r of the system is equal to the ts-QHWFT of the input signal f on the point $(\underline{\mathbf{b}}, \mathbf{b})$ of the phase-space.

Proof. Replacing h_1 defined in (48) in Definition 5.1, and using the fact that $f\mathbf{i} = \mathbf{i}f$, we have

$$\begin{aligned} r_1(\mathbf{b}) &= \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi \mathbf{i} \underline{b}_1 (\underline{b}_1 \ominus (\underline{b}_1 \ominus \underline{x}_1))} f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) e^{-2\pi \mathbf{j} \underline{b}_2 (\underline{b}_2 \ominus (\underline{b}_2 \ominus \underline{x}_2))} d\mu(\mathbf{x}) \\ &= \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi \mathbf{i} \underline{b}_1 \underline{x}_1} f(\mathbf{x}) g(\mathbf{x} \ominus \mathbf{b}) e^{-2\pi \mathbf{j} \underline{b}_2 \underline{x}_2} d\mu(\mathbf{x}) \\ &= \mathcal{Q}_g f(\underline{\mathbf{b}}, \mathbf{b}). \end{aligned}$$

□

For another choice of impulse response filter we can relate the output response of the SVHS with the ts-QHFT.

Theorem 5.2. Consider a SVHS band-pass filter with the quaternion impulse response

$$h_2(\mathbf{b}, \mathbf{x}) = e^{-2\pi \mathbf{i} \underline{b}_1 (\underline{b}_1 \ominus \underline{x}_1)} e^{-2\pi \mathbf{j} \underline{b}_2 (\underline{b}_2 \ominus \underline{x}_2)}, \quad \mathbf{b}, \mathbf{x} \in \mathbb{R}_{t_1, t_2}^2. \quad (49)$$

If the input to this system is the hyperbolic quaternion signal f obeying $f\mathbf{i} = \mathbf{i}f$, then its output is equal to the ts-QHFT, i.e.,

$$r_2(\mathbf{b}) = \int_{\mathbb{R}_{t_1, t_2}^2} f(\mathbf{x}) h_2(\mathbf{b}, \mathbf{b} \ominus \mathbf{x}) d\mu(\mathbf{x}) = \mathcal{F}_{QH}(f)(\underline{\mathbf{b}}).$$

Example 5.1. Given the SVHS filter (49) and the signal f defined by

$$f(\mathbf{x}) = \begin{cases} e^{-\underline{x}_1 + \underline{x}_2}, & \text{if } (x_1, x_2) \in [0, t_1) \times [0, t_2) \\ 0, & \text{otherwise} \end{cases}$$

then from Theorem 5.2 the output $r_2(\cdot)$ of f is given by

$$\begin{aligned}
r_2(\mathbf{b}) &= \int_{\mathbb{R}_{t_1, t_2}^2} e^{-2\pi \mathbf{i} \underline{b}_1 x_1} e^{-(x_1 + x_2)} e^{-2\pi \mathbf{j} \underline{b}_2 x_2} d\mu(\mathbf{x}) \\
&= \int_0^{t_1} \int_0^{t_2} e^{-x_1(1-2\pi \mathbf{i} \underline{b}_1)} e^{-x_2(1-2\pi \mathbf{j} \underline{b}_2)} d\mu(\mathbf{x}) \\
&= \frac{-1}{1-2\pi \mathbf{i} \underline{b}_1} \frac{-1}{1-2\pi \mathbf{j} \underline{b}_2} \\
&= \frac{1}{1-2\pi \mathbf{i} \underline{b}_1 - 2\pi \mathbf{j} \underline{b}_2 + 4\pi^2 \underline{b}_1 \underline{b}_2 \mathbf{k}} \\
&= \frac{1 + 2\pi \mathbf{i} \underline{b}_1 + 2\pi \mathbf{j} \underline{b}_2 - 4\pi^2 \underline{b}_1 \underline{b}_2 \mathbf{k}}{1 + (2\pi \underline{b}_1)^2 + (2\pi \underline{b}_2)^2 + (4\pi^2 \underline{b}_1 \underline{b}_2)^2}.
\end{aligned}$$

6. Conclusion

In this work, we introduced the hyperbolic counterpart or the two-sided QWFT for quaternion-valued signals defined in an open rectangle of the Euclidean plane, equipped with a hyperbolic measure. The main properties and theorems associated with the two-sided QHWFT are given, including modulation, hyperbolic translation, hyperbolic dilation, Moyal's identity, and an inversion formula. By introducing the relativistic counterpart of the quaternion Wiener space and using the quaternion hyperbolic Poisson summation formula deduced in this paper, we obtained Walnut's and Janssen's representation theorems for the quaternion hyperbolic frame operator, as well as quaternionic versions of the Wexler-Raz biorthogonality and Ron-Shen duality. A thorough investigation of the characterization of quaternionic hyperbolic tight Gabor frames is also presented. As an application, we demonstrate the use of the ts-QHWFT in analyzing two-dimensional linear space-varying hyperbolic systems that illustrate the applicability of our results. The results of this work can be extended to encompass quaternionic window functions, as was done in [32].

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